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ARITHMETIC MEAN FUNCTION OF THE REAL PART OF ENTIRE DIRICHLET SERIES (II)

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1. Let E be the set of mappings $f: C \to C$ (C is the complex plane) such that the image under f of a point $s \in C$ is $f(s) = \sum_{n \in N} a_n e^{s\lambda_n}$ with $\limsup_{n\to+\infty} \frac{\log n}{\lambda_n} = D \in R_+ \cup \{0\}$ (R_+ is the set of positive reals), and $\sigma_c^f = +\infty$ (σ_c^f is the abscissa of convergence of the Dirichlet series defining f) ; N is the set of natural numbers 0, 1, $\cdots$, $\langle a_n | n \in N \rangle$ is a sequence in C, $s = \sigma + it$, σ, $t \in R$ (R is the field of reals, and $\langle \lambda_n | n \in N \rangle$ is a strictly increasing unbounded sequence of nonnegative reals. Since the Dirichlet series defining f converges for each $s \in C$, f is an entire function. Also, since $D \in R_+ \cup \{0\}$, we have ([1], p.168) $\sigma_a^f = +\infty$ (σ_a^f is the abscissa of absolute convergence of the Dirichlet series defining f) and that f is bounded on each vertical line $\text{Re}(s) = \sigma_0$.

Let

$$M(\sigma, f) = \sup_{t \in R} \{|f(\sigma+it)|\}, \ \forall\, \sigma < \sigma_c^f,$$

be the maximum modulus of an entire function $f \in E$ on any vertical line $\text{Re}(s) = \sigma$,

$$\mu(\sigma, f) = \max_{n \in N} \{|a_n| e^{\sigma\lambda_n}\}, \ \forall \sigma < \sigma_c^f,$$

be the maximum term, for $\text{Re}(s) = \sigma$, in the Dirichlet series defining f, and

$$\nu(\sigma, f) = \max_{n \in N} \{n \,|\, \mu(\sigma, f) = |a_n| e^{\sigma\lambda_n}\}, \ \forall \sigma < \sigma_c^f,$$

be the rank of the maximum term.

The arithmetic mean function A and the generalized arithmetic mean function J_r of $\text{Re}(f)$ are defined [2], respectively, as

$$A(\sigma, f) = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} |\text{Re}(f(\sigma+it))|\, dt, \ \forall\, \sigma < \sigma_c^f, \tag{1.1}$$

and

$$J_r(\sigma, f) = \lim_{T\to+\infty} \frac{1}{2Te^{r\sigma}} \int_0^{\sigma} \int_{-T}^{T} |\text{Re}(f(x+it))| e^{rx}\, dx\, dt, \ \forall \sigma < \sigma_c^f. \tag{1.2}$$

We shall denote by $J_{r,p}$, $\forall p \in Z_+$ (Z_+ is the set of positive integers), the generalized arithmetic mean function of the real part of the p-th derivative $f^{(p)}$ of f.

In our earlier paper [2] we studied some properties of the functions A and J_r; in this paper we study a few more properties of these functions.

2. THEOREM 1. *For every entire function $f \in E$ of Ritt order $\rho \in R_+^* \cup \{0\}$ (R_+^* is the set of extended positive reals) and lower order $\lambda \in R_+^* \cup \{0\}$, if, for any $\varepsilon \in R_+$, $\lambda_{\nu(\sigma,f)} \sim \lambda_{\nu(\sigma+D+\varepsilon,f)}$ as $\sigma \to +\infty$, then*

$$(2.1)\quad \liminf_{\sigma \to +\infty} \frac{\log J_r(\sigma,f)}{\lambda_{\nu(\sigma,f)}} \leq \frac{1}{\rho} \leq \frac{1}{\lambda} \leq \limsup_{\sigma \to +\infty} \frac{\log J_r(\sigma,f)}{\lambda_{\nu(\sigma,f)}}.$$

PROOF. It is known ([2], Formula (2.7)) that

$$\mu(\sigma,f) \leq 4A(\sigma,f) \leq 4M(\sigma,f),$$

and ([3], p. 68) that, for any $\varepsilon \in R_+$ and $\sigma > \sigma_0(\varepsilon,f)$,

$$M(\sigma,f) < \mu(\sigma+D+\varepsilon,f).$$

Therefore, for any $\varepsilon \in R_+$ and $\sigma > \sigma_0(\varepsilon,f)$,

$$\mu(\sigma,f) \leq 4A(\sigma,f) < 4\mu(\sigma+D+\varepsilon,f),$$

and hence

$$(2.2)\quad \lim_{\sigma \to +\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(\sigma,f)}{\lambda_{\nu(\sigma,f)}} = \lim_{\sigma \to +\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log A(\sigma,f)}{\lambda_{\nu(\sigma,f)}}.$$

But ([4], p. 87),

$$(2.3)\quad \liminf_{\sigma \to +\infty} \frac{\log \mu(\sigma,f)}{\lambda_{\nu(\sigma,f)}} \leq \frac{1}{\rho} \leq \frac{1}{\lambda} \leq \limsup_{\sigma \to +\infty} \frac{\log \mu(\sigma,f)}{\lambda_{\nu(\sigma,f)}}.$$

From (2.2) and (2.3) it, therefore, follows that

$$(2.4)\quad \liminf_{\sigma \to +\infty} \frac{\log A(\sigma,f)}{\lambda_{\nu(\sigma,f)}} \leq \frac{1}{\rho} \leq \frac{1}{\lambda} \leq \limsup_{\sigma \to +\infty} \frac{\log A(\sigma,f)}{\lambda_{\nu(\sigma,f)}}.$$

Now, from (1.1) and (1.2), we have

$$(2.5)\quad J_r(\sigma,f) = \frac{1}{e^{r\sigma}} \int_0^\sigma A(x,f)\, e^{rx}\, dx$$

$$(2.6)\quad \leq A(\sigma,f) \frac{1}{r}(1-e^{-r\sigma}),$$

since ([2], Theorem 1) A is an increasing function of σ. Therefore

$$(2.7)\quad \liminf_{\sigma \to +\infty} \frac{\log J_r(\sigma,f)}{\lambda_{\nu(\sigma,f)}} \leq \liminf_{\sigma \to +\infty} \frac{\log A(\sigma,f)}{\lambda_{\nu(\sigma,f)}}.$$

Also, from (2.5), for any $h \in R_+$,

$$J_r(\sigma+h, f) \geq \frac{1}{e^{r(\sigma+h)}} \int_{\sigma}^{\sigma+h} A(x, f) e^{rx}\, dx$$

$$\geq A(\sigma, f) \frac{1}{r}(1-e^{-rh}),$$

and so

$$\limsup_{\sigma\to+\infty} \frac{\log J_r(\sigma, f)}{\lambda_{\nu(\sigma,f)}} \geq \limsup_{\sigma\to+\infty} \frac{\log A(\sigma, f)}{\lambda_{\nu(\sigma,f)}}. \tag{2.8}$$

(2.1) now follows from (2.4), (2.7) and (2.8).

THEOREM 2. *For every entire function $f \in E$ of Ritt order $\rho \in R_+$ and lower order $\lambda \in R_+$,*

$$\limsup_{\sigma\to+\infty} \frac{\log J_r(\sigma, f)}{\sigma \lambda_{\nu(\sigma,f)}} \leq 1 - \frac{\lambda}{\rho}, \tag{2.9}$$

and

$$\limsup_{\sigma\to+\infty} \frac{\log J_r(\sigma, f)}{\lambda_{\nu(\sigma,f)} \log \lambda_{\nu(\sigma,f)}} \leq \frac{1}{\lambda} - \frac{1}{\rho}. \tag{2.10}$$

PROOF. We have, from (2.6),

$$\limsup_{\sigma\to+\infty} \frac{\log J_r(\sigma, f)}{\sigma \lambda_{\nu(\sigma,f)}} \leq \limsup_{\sigma\to+\infty} \frac{\log A(\sigma, f)}{\sigma \lambda_{\nu(\sigma,f)}}$$

$$= \limsup_{\sigma\to+\infty} \frac{\log \mu(\sigma, f)}{\sigma \lambda_{\nu(\sigma,f)}}, \text{ in view of (2.2)}$$

$$\leq 1 - \frac{\lambda}{\rho}, \text{ in view of ([5], Formula (4.2)).}$$

Thus (2.9) is established. Using ([5], Formula (4.11)), (2.10) can also be established similarly.

THEOREM 3. *For every entire function $f \in E$, if $\sigma_1, \sigma_2 \in R_+$ are such that $0 < \sigma_1 < \sigma_2 < \sigma_c^f$, then*

$$A(\sigma_1, f) \leq r\left(\frac{e^{r\sigma_2} J_r(\sigma_2, f) - e^{r\sigma_1} J_r(\sigma_1, f)}{e^{r\sigma_2} - e^{r\sigma_1}}\right) \leq A(\sigma_2, f). \tag{2.11}$$

PROOF. We have, from (2.5),

$$J_r(\sigma_1, f) = \frac{1}{e^{r\sigma_1}} \int_0^{\sigma_1} A(x, f) e^{rx}\, dx,$$

and

$$J_r(\sigma_2,f)=\frac{1}{e^{r\sigma_2}}\int_0^{\sigma_2} A(x,f)\ e^{rx}\ dx.$$

Therefore

$$e^{r\sigma_2}J_r(\sigma_2,f)-e^{r\sigma_1}J_r(\sigma_1,f)=\int_{\sigma_1}^{\sigma_2} A(x,f)\ e^{rx}\ dx. \tag{2.12}$$

From (2.12) it follows that

$$e^{r\sigma_2}\ J_r(\sigma_2,f)-e^{r\sigma_1}\ J_r(\sigma_1,f)\le A(\sigma_2,f)\frac{1}{r}(e^{r\sigma_2}-e^{r\sigma_1}), \tag{2.13}$$

and

$$e^{r\sigma_2}\ J_r(\sigma_2,f)-e^{r\sigma_1}\ J_r(\sigma_1,f)\ge A(\sigma_1,f)\frac{1}{r}(e^{r\sigma_2}-e^{r\sigma_1}). \tag{2.14}$$

Combining (2.13) and (2.14), we get (2.11).

THEOREM 4. *For every entire function $f\in E$, and sufficiently large σ,*

$$J_{r,1}(\sigma,f^{(1)})\ge J_r(\sigma,f)\ \frac{\log J_r(\sigma,f)}{\sigma}. \tag{2.15}$$

PROOF. We have, from (1.2),

$$J_{r,1}(\sigma,f^{(1)})=\lim_{T\to+\infty}\frac{1}{2Te^{r\sigma}}\int_0^{\sigma}\int_{-T}^{T}|\mathrm{Re}(f^{(1)}(x+it))|e^{rx}\ dx\ dt$$

$$=\lim_{T\to+\infty}\frac{1}{2Te^{r\sigma}}\int_0^{\sigma}\int_{-T}^{T}\left|\lim_{h\to0}\frac{\mathrm{Re}(f(x+it))-\mathrm{Re}(f(x-h+it))}{h}\right|e^{rx}\ dx\ dt$$

$$\ge\lim_{T\to+\infty}\frac{1}{2Te^{r\sigma}}\int_0^{\sigma}\int_{-T}^{T}\lim_{h\to0}\frac{|\mathrm{Re}(f(x+it))|-|\mathrm{Re}(f(x-h+it))|}{h}\ e^{rx}\ dx\ dt$$

$$=\lim_{h\to0}\lim_{T\to+\infty}\frac{1}{2Te^{r\sigma}}\int_0^{\sigma}\int_{-T}^{T}\frac{|\mathrm{Re}(f(x+it))|-|\mathrm{Re}(f(x-h+it))|}{h}\ e^{rx}\ dx\ dt$$

$$=\lim_{h\to0}\left(\frac{J_r(\sigma,f)-J_r(\sigma-h,f)}{h}\right).$$

Let

$$\phi(\sigma)=\frac{\log J_r(\sigma,f)}{\sigma}\ ;$$

then, for sufficiently large σ, ϕ is an increasing function of σ. Therefore

$$J_{r,1}(\sigma,f)\geq\lim_{h\to0}\left(\frac{e^{\sigma\phi(\sigma,f)}-e^{(\sigma-h)\phi(\sigma-h,f)}}{h}\right)$$
$$\geq\phi(\sigma,f)e^{\sigma\phi(\sigma,f)}$$
$$=\frac{\log J_r(\sigma,f)}{\sigma}J_r(\sigma,f).$$

Hence the result.

Following is an interesting corollary to the above theorem.

COROLLARY 1. *If f is of Ritt order $\rho\in R_+{}^*\cup\{0\}$ and lower order $\lambda\in R_+{}^*\cup\{0\}$, then*

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log(J_{r,1}(\sigma,f^{(1)})/J_r(\sigma,f))}{\sigma}\geq\begin{matrix}\rho\\ \lambda\end{matrix}. \tag{2.16}$$

But if $\lambda\geq\sigma>0$, then, for sufficiently large σ,

$$J_r(\sigma,f)<J_{r,1}(\sigma,f^{(1)})<\cdots<J_{r,p}(\sigma,f^{(p)})<\cdots, \tag{2.17}$$

and

$$J_{r,p}(\sigma,f^{(p)})>J_r(\sigma,f)\left(\frac{\log J_r(\sigma,f)}{\sigma}\right)^p. \tag{2.18}$$

PROOF. The proof of (2.16) follows from (2.15) and the fact ([2], Theorem 2) that

$$\begin{matrix}\rho\\ \lambda\end{matrix}=\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log\log J_r(\sigma,f)}{\sigma}. \tag{2.19}$$

In order to prove (2.17) we write (2.16) for the m-th derivative $f^{(m)}$ of f and get

$$\lim_{\sigma\to+\infty}\begin{matrix}\sup\\ \inf\end{matrix}\frac{\log(J_{r,m}(\sigma,f^{(m)})/J_{r,m-1}(\sigma,f^{(m-1)})}{\sigma}\geq\begin{matrix}\rho\\ \lambda\end{matrix}.$$

Hence, for any $\varepsilon\in R_+$ and sufficiently large σ,

$$J_{r,m-1}(\sigma,f^{(m-1)})\,e^{\sigma(\lambda-\varepsilon)}<J_{r,m}(\sigma,f^{(m)}).$$

Since $\lambda\geq\sigma>0$ and ε is arbitrary, it follows that, for sufficiently large σ,

$$J_{r,m-1}(\sigma,f^{(m-1)})<J_{r,m}(\sigma,f^{(m)}).$$

Giving m the values 1, 2, $\cdots$, p, $\cdots$, and combining the resulting inequalities we get (2.17).

Finally, for the proof of (2.18), we write (2.15) for the m-th derivative obtaining, for sufficiently large σ,

$$\frac{J_{r,m}(\sigma,f^{(m)})}{J_{r,m-1}(\sigma,f^{(m-1)})} \geq \frac{\log J_{r,m-1}(\sigma,f^{(m-1)})}{\sigma}. \tag{2.20}$$

Putting $m=1, 2, \cdots, p$ in (2.20) and multiplying the p inequalities thus obtained we get, for sufficiently large σ.

$$\frac{J_{r,p}(\sigma,f^{(p)})}{J_r(\sigma,f)} \geq \frac{\prod_{m=1}^{p} \log J_{r,m-1}(\sigma,f^{(m-1)})}{\sigma^p},$$

which in view of (2.17), gives (2.18).

The next corollary is immediate from (2.18) and (2.19).

COROLLARY 2. *For every entire function $f \in E$ of Ritt order $\rho \in R_+$ and lower order $\lambda \in R_+$ such that $\lambda \geq \delta > 0$,*

$$\lim_{\sigma\to+\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log (J_{r,p}(\sigma,f^{(p)})/J_r(\sigma,f))^{1/p}}{\sigma} \geq \begin{matrix}\rho\\ \lambda\end{matrix}, \quad \forall p \in Z_+. \tag{2.21}$$

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