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THE NECESSARY CONDITIONS FOR CONFORMALLY FLAT SPACES TO BE SPACE FORMS

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1. Introduction.

A Reimannian manifold (M, g) of dimension n is said to be a conformally flat space if a Riemannian metric g is conformally related to a Riemannian metric g^* which is locally Euclidean.

The tensor field C defined by

(0.1) $$C_{kji}{}^h = K_{kji}{}^h + L_k{}^h g_{ji} - L_j{}^h g_{ki} + \delta_k^h L_{ji} - \delta_j^h L_{ki},$$

where

(0.2) $$L_{ji} = -\frac{1}{n-2} K_{ji} + \frac{K}{2(n-1)(n-2)} g_{ji},$$

is called the Weyl conformal curvature tensor, $K_{kji}{}^h$, K_{ji} and K denoting curvature tensor, Ricci tensor and scalar curvature of (M, g) respectively.

Weyl([7], [8]) characterized a conformally flat space as follows:

An n-dimensional Riemannian manifold (M, g) is conformally flat if and only if

(0.3) $$C_{kji}{}^h = 0 \text{ for } n > 3,$$

(0.4) $$\nabla_k L_{ji} - \nabla_j L_{ki} = 0 \text{ for } n = 3,$$

where ∇_j the operator of covariant differentiation with respect to the Christoffel symbols $\{{}_j{}^h{}_i\}$.

It is well known that (0.4) can be derived from (0.3) for $n > 3$.

Recently Ryan [4] studied a compact conformally flat space with constant scalar curvature and with positive semi-definite Ricci tensor (Also [1], [6]).

On the other hand, Yano, Houh and Chen discussed in their paper [10] an intrinsic problem of a conformally flat space in terms of the sectional curvature with respect to certain unit vector field. Moreover, Chen and Yano [2] investigated a necessary and sufficient condition for a Riemannian manifold of dimension $n > 3$ with mutually orthogonal unit vector fields satisfying some algebraic conditions to be conformally flat.

In the present paper, we study conformally flat spaces with various types of conditions of sectional curvatures and find necessary conditions that allow a conformally flat space with mutually orthogonal unit vector fields to be a space form.

In section 1, we first prove that if a Riemannian manifold (M, g) of dimension $n > 3$ with a unit vector field u^h satisfies certain algebraic conditions, then (M, g) is conformally flat. Moreover we prove that such a complete conformally flat space is decomposed into one of $S^n(c)$, $R \times S^{n-1}(c)$ or E^n, the real space forms of curvature c being denoted by $S^n(c)$ or E^n depending on whether c is positive or zero.

In the last section 2, we define a *special conformally flat space*, that is, a conformally flat space with two mutually orthogonal unit vector fields u^h and v^h such that

$$L_{ji} = \alpha g_{ji} + \beta(u_j v_i + v_j u_i),$$

where α and β are functions.

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We suppose that the vector field u^h is parallel along geodesics in a special conformally flat space. Then the space becomes a space form by establishing three important lemmas prepared before the main theorem.

Finally, we can also prove the same result as that stated in the main theorem, under the assumption of the compactness without parallelism of the vector field u^h along geodesics.

1. Characterizations of conformally flat spaces

We prepare a lemma for later use.

LEMMA 1.1. *In a conformally flat space of dimension n with constant scalar curvature K, we have* (*cf.* [11])

$$\frac{1}{2}\Delta(K_{ji}K^{ji})=\frac{n}{n-2}K_t{}^sK_s{}^rK_r{}^t-\frac{2n-1}{(n-1)(n-2)}KK_{ji}K^{ji}+\frac{1}{(n-1)(n-2)}K^3+(\nabla_jK_{ih})(\nabla^jK^{ih}). \tag{1.1}$$

In this section, we would like to consider intrinsic characterizations of a Riemannian manifold (M^n, g) with the curvature tensor $K_{kji}{}^h$ given in the form of (1.3).

Yano, Houh and Chen showed the following in [10]

THEOREM A. *Let (M^n, g) be an n-dimensional Riemannian manifold with a unit vector field ξ^h. Then the necessary and sufficient conditions for (M^n, g) having the properties:*

(I) *The curvature operator $K_{kji}{}^hv^kw^j$ associated with two vectors v^h and w^h orthogonal to ξ^h annihilates ξ^h:*

$$K_{kjih}v^kw^j\xi^i=0; \tag{1.2}$$

(II) *Sectional curvature with respect to a section containing ξ^h is a constant;*

(III) *Sectional curvature with respect to a section orthogonal to ξ^h is a constant;*

is that the Riemannian curvature tensor of (M^n, g) has the form

$$K_{kji}{}^h=\lambda(\delta_k^hg_{ji}-\delta_j^hg_{ki})+\mu\{(\delta_k^h\xi_j-\delta_j^h\xi_k)\xi_i+(\xi_kg_{ji}-\xi_jg_{ki})\xi^h\} \tag{1.3}$$

for some functions λ and μ. In this case, (M^n, g) is a conformally flat space for $n>3$.

We first prove

THEOREM 1.2. *Let (M^n, g) be an n-dimensional Riemannian manifold with a unit vector field ξ^h. Then the necessary and sufficient conditions for (M^n, g) having properties* (I), (III) *and the Weyl conformal curvature tensor $C_{kji}{}^h$ satisfying*

$$C_{kji}{}^h\xi^j\xi^i=0 \tag{1.4}$$

is that the Riemannian curvature tensor of (M^n, g) has the form of (1.3) *for $n>3$.*

Proof. We take $n-1$ linearly independent vectors $B_b{}^h$ $(a, b, c, \cdots$ run over the range $1, 2, \cdots, n-1)$ orthogonal to ξ^h and let $B^a{}_i$, ξ_i be determined in such a way that $(B_b{}^h, \xi^h)^{-1}=(B^a{}_i, \xi_i)$. Then we have

$$B_a{}^hB^a{}_i=\delta_i^h-\xi_i\xi^h.$$

The condition (I) is expressed as

$$B_e{}^k B_d{}^j K_{kji}{}^h \xi^i = 0.$$

Transvecting $B^e{}_m B^d{}_l$ to this equation we find

$$(\delta_m{}^k - \xi_m \xi^k)(\delta_l{}^j - \xi_l \xi^j) K_{kji}{}^h \xi^i = 0$$

or, equivalently

$$K_{mli}{}^h \xi^i - K_{mji}{}^h \xi^i \xi_l \xi^j - K_{kli}{}^h \xi^k \xi_m \xi^i = 0,$$

from which

(1.5) $$K_{kji}{}^h \xi^i = M_k{}^h \xi_j - M_j{}^h \xi_k,$$

where we have put

(1.6) $$M_k{}^h = K_{kji}{}^h \xi^j \xi^i.$$

$M_k{}^h$ satisfies

(1.7) $$M_k{}^h \xi_h = 0, \qquad M_k{}^h \xi^k = 0, \qquad M_{ji} = M_{ij},$$

where $M_{ji} = M_j{}^t g_{ti}$.

From the condition (Ⅲ) we have

$$K_{kjih} B_d{}^k B_c{}^j B_b{}^i B_a{}^h = k(g_{da} g_{cb} - g_{ca} g_{db}),$$

Transvecting this equation with $B^d{}_s B^c{}_r B^b{}_q B^a{}_p$ we find

$$K_{kjih}(\delta_s{}^k - \xi_s \xi^k)(\delta_r{}^j - \xi_r \xi^j)(\delta_q{}^i - \xi_q \xi^i)(\delta_p{}^h - \xi_p \xi^h)$$
$$= k\{(g_{sp} - \xi_s \xi_p)(g_{rq} - \xi_r \xi_q) - (g_{rp} - \xi_r \xi_p)(g_{sq} - \xi_s \xi_q)\},$$

or, using (1.7)

$$K_{srqp} - \xi_p K_{srqt} \xi^t - \xi_q K_{srtp} \xi^t - \xi_r K_{stqp} \xi^t - \xi_s K_{trqp} \xi^t$$
$$- M_{sq} \xi_r \xi_p + M_{sp} \xi_q \xi_r - M_{rp} \xi_s \xi_q + M_{rq} \xi_s \xi_p$$
$$= k\{g_{sp} g_{rq} - g_{rp} g_{sq} - (g_{sp} \xi_r \xi_q - g_{rp} \xi_s \xi_q) - (g_{rq} \xi_s \xi_p - g_{sq} \xi_r \xi_p)\},$$

from which, substituting (1.5) and using the last equation of (1.7),

$$K_{srqp} + (M_{sq} \xi_r \xi_p - M_{sp} \xi_r \xi_q) - (M_{qr} \xi_p \xi_s - M_{pr} \xi_q \xi_s)$$
$$= k\{g_{sp} g_{rq} - g_{rp} g_{sq} - (g_{sp} \xi_r \xi_q - g_{rp} \xi_s \xi_q) - (g_{rq} \xi_s \xi_p - g_{sq} \xi_r \xi_p)\},$$

that is,

(1.8) $$K_{kjih} = k(g_{kh} g_{ji} - g_{jh} g_{ki}) + (M_{kh} - k g_{kh}) \xi_j \xi_i - (M_{jh} - k g_{jh}) \xi_k \xi_i + (M_{ji} - k g_{ji}) \xi_k \xi_h - (M_{ki} - k g_{ki}) \xi_j \xi_h.$$

Transvecting (1.8) with g^{kh} and using (1.7), we obtain the relationship

(1.9) $$K_{ji} = k(n-2) g_{ji} + \{M_t{}^t - k(n-2)\} \xi_j \xi_i + M_{ji},$$

from which

(1.10) $$K = k(n-1)(n-2) + 2M_t{}^t.$$

Substituting (1.9) and (1.10) into (0.2), we find

(1.11) $$L_{ji} = -\frac{1}{n-2}[k(n-2) g_{ji} + \{M_t{}^t - k(n-2)\} \xi_j \xi_i + M_{ji}] + \frac{k(n-1)(n-2) + 2M_t{}^t}{2(n-1)(n-2)} g_{ji}$$
$$= \left\{ \frac{1}{(n-1)(n-2)} M_t{}^t - \frac{1}{2} k \right\} g_{ji} + \left(k - \frac{1}{n-2} M_t{}^t\right) \xi_j \xi_i - \frac{1}{n-2} M_{ji}.$$

If we substitute (1.8) and (1.11) into (0.1), then

(1.12)
$$C_{kjih}=\frac{2}{(n-1)(n-2)}M_t^t(g_{kh}g_{ji}-g_{jh}g_{ki})$$
$$-\frac{1}{n-2}(g_{kh}M_{ji}-g_{jh}M_{ki}+g_{ji}M_{kh}-g_{ki}M_{jh})$$
$$+(M_{kh}\xi_j-M_{jh}\xi_k)\xi_i+(M_{ji}\xi_k-M_{ki}\xi_j)\xi_h$$
$$-\frac{1}{n-2}M_t^t\{(g_{kh}\xi_j-g_{jh}\xi_k)\xi_i+(g_{ji}\xi_k-g_{ki}\xi_j)\xi_h\}.$$

Transvecting $\xi^k\xi^h$ to (1.12) and making use of the condition $C_{kjih}\xi^k\xi^h=0$, we get
$$\frac{2}{(n-1)(n-2)}M_t^t(g_{ji}-\xi_j\xi_i)+M_{ji}-\frac{1}{n-2}M_t^t(g_{ji}-\xi_j\xi_i)=0$$
by virtue of (1.7), or equivalently,
$$\frac{n-3}{n-2}\left\{M_{ji}-\frac{1}{n-1}M_t^t(g_{ji}-\xi_j\xi_i)\right\}=0.$$
Since $n>3$, M_{ji} has the form

(1.13)
$$M_{ji}=\lambda(g_{ji}-\xi_j\xi_i),$$
where $\lambda=\frac{1}{n-1}M_t^t$.

Substituting (1.13) into (1.8), we have

(1.14)
$$K_{kji}{}^h=k(\delta_k^h g_{ji}-\delta_j^h g_{ki})+(\lambda-k)\{(\delta_k^h\xi_j-\delta_j^h\xi_k)\xi_i+(g_{ji}\xi_k-g_{ki}\xi_j)\xi^h\}.$$
This is the form of (1.3). Hence Theorem 1.2 is proved.

Combining Theorem A and Theorem 1.2, we have

COROLLARY 1.3. *Under the same assumptions as those stated in Theorem* 1.2, *the Riemannian manifold* (M^n, g) *is conformally flat.*

Finally we prove

THEOREM 1.4. *Let* (M^n, g) *be a complete Riemannian manifold* $(n>3)$ *such that the conditions* (I), (II) *and* (III) *of Theorem A are satisfied at every point of* (M^n, g). *Then* (M^n, g) *is one of*
$$S^n(c), \qquad R\times S^{n-1}(c) \qquad or \qquad E^n,$$
the real space forms of curvature c being denoted by $S^n(c)$ *or* E^n *depending on whether c is positive or zero (cf. Ryan* [4], *Goldberg* [3], *Tani* [6], *Sekigawa and Takagi* [5]).

Proof. By assumptions, equation (1.4) is valid, and consequently

(1.15)
$$K_{ji}=\{(n-2)k+\lambda\}g_{ji}+(\lambda-k)(n-2)\xi_j\xi_i,$$

(1.16)
$$K=(n-1)\{(n-2)k+2\lambda\}.$$

The conditions (II) and (III) mean that λ and k are both constants, which imply that the scalar curvature K is constant. Thus we see that

(1.17)
$$\nabla_k K_{ji}-\nabla_j K_{ki}=0$$
by virtue of (0.2) and (0.3). We have from (1.15)

(1.18)
$$K_{jt}\xi^t=(n-1)\lambda\xi_j$$
and

(1.19) $$K_{jt}K_i^{\ t}=\{(n-2)k+\lambda\}K_{ji}+(n-1)(n-2)\lambda(\lambda-k)\xi_j\xi_i$$

from which, transvecting with g^{ji},

(1.20) $$K_{ji}K^{ji}=K\{(n-2)k+\lambda\}+(n-1)(n-2)\lambda(\lambda-k).$$

Transvecting (1.19) with K^{ji} and using (1.18) and (1.20), we find

(1.21) $$K_t^{\ s}K_s^{\ r}K_r^{\ t}=[(n-2)k+\lambda][K\{(n-2)k+\lambda\}+(n-1)(n-2)\lambda(\lambda-k)] +(n-1)^2(n-2)\lambda^2(\lambda-k).$$

Differentiating (1.15) covariantly and taking account of the constancies of λ and k, we obtain

(1.22) $$\nabla_k K_{ji}=(n-2)(\lambda-k)\{(\nabla_k\xi_j)\xi_i+\xi_j(\nabla_k\xi_i)\},$$

from which, using (1.17)

$$(\lambda-k)\{(\nabla_k\xi_j-\nabla_j\xi_k)\xi_i+\xi_j\nabla_k\xi_i-\xi_k\nabla_j\xi_i\}=0.$$

Transvecting ξ^j to this equation, we find

(1.23) $$(\lambda-k)\{-(\xi^t\nabla_t\xi_k)\xi_i+\nabla_k\xi_i-\xi_k(\xi^t\nabla_t\xi_i)\}=0,$$

from which, transvecting with ξ^i,

$$(\lambda-k)\xi^t\nabla_t\xi_k=0.$$

Thus (1.23) reduces to $(\lambda-k)\nabla_k\xi_i=0$ and consequently (1.22) becomes

(1.24) $$\nabla_k K_{ji}=0.$$

Therefore. (1.1) becomes

$$nK_t^{\ s}K_s^{\ r}K_r^{\ t}-\frac{2n-1}{n-1}KK_{ji}K^{ji}+\frac{1}{n-1}K^3=0.$$

Substituting (1.20) and (1.21) into the equation above, we have

$$n[(n-2)k+\lambda][K\{(n-2)k+\lambda\}+(n-1)(n-2)\lambda(\lambda-k)] +n(n-1)^2(n-2)\lambda^2(\lambda-k)-\frac{2n-1}{n-1}K^2\{(n-2)k+\lambda\} -(2n-1)(n-2)\lambda(\lambda-k)K+\frac{1}{n-1}K^3=0,$$

or using (1.16), $(n-1)(n-2)^3\lambda(\lambda-k)^2=0$, which implies that

$$\lambda(\lambda-k)=0 \quad \text{i.e.,} \quad \lambda=0 \quad \text{or} \quad \lambda=k$$

because λ and k are both constants. If $\lambda=k$, then (M^n, g) is a space form by virtue of (1.14). We consider only the case which $\lambda=0$. In this case (1.19) becomes

(1.25) $$K_{jt}K_i^{\ t}=(n-2)kK_{ji}.$$

Let ρ be an arbitrary eigenvalue of $K_i^{\ h}$ associated with an eigenvector of $K_i^{\ h}$. Then using (1.25), we see that ρ satisfies the quadratic equation $\rho^2-(n-2)k\rho=0$. Thus $K_i^{\ h}$ has at most two constant eigenvalues 0 and $(n-2)k$. Taking account of (1.16) with $\lambda=0$, the multiplicity of the eigenvalue 0 is 1.

In usual way (cf. Ryan [4], Sekigawa and Takagi [5]), owing to completeness, (M^n, g) is one of

$$R\times S^{n-1}(c) \quad \text{if} \quad k\neq 0$$

or

$$E^n \quad \text{if} \quad k=0.$$

This completes the proof of Theorem 1.4.

2. Special conformally flat spaces with constant scalar curvature.

A Riemannian manifold (M, g) of dimension n is said to *be special conformally flat* if there exist, on conformally flat space (M, g), two mutually orthogonal unit vector fields u^h and v^h such that

$$L_{ji}=\alpha g_{ji}+\beta(u_jv_i+v_ju_i) \tag{2.1}$$

for any functions α and β.

Throughout this section, we suppose that a Riemannian manifold (M, g) of dimension $n>2$ is special conformally flat. Then we have from (0.1) and (0.2)

$$K_{kjih}=-L_{kh}g_{ji}+L_{ki}g_{jh}-L_{ji}g_{kh}+L_{jh}g_{ki}, \tag{2.2}$$

where

$$L_{ji}=-\frac{1}{n-2}K_{ji}+\frac{K}{2(n-1)(n-2)}g_{ji}, \tag{2.3}$$

from which, substituting (2.1)

$$K_{kjih}=2\alpha(g_{jh}g_{ki}-g_{kh}g_{ji})+\beta\{g_{jh}(u_kv_i+v_ku_i)-g_{ji}(u_kv_h+v_ku_h)$$
$$+g_{ki}(u_jv_h+v_ju_h)-g_{kh}(u_jv_i+v_ju_i)\}, \tag{2.4}$$

$$-\frac{1}{n-2}K_{ji}=\frac{2(n-1)}{n-2}\alpha g_{ji}+\beta(u_jv_i+v_ju_i). \tag{2.5}$$

Transvecting (2.5) with g^{ji}, we have

$$K=-2n(n-1)\alpha. \tag{2.6}$$

Transvecting (2.5) with u^i and v^i respectively and using (2.6), we find

$$K_{jt}u^t=\frac{K}{n}u_j-(n-2)\beta v_j, \tag{2.7}$$

$$K_{jt}v^t=\frac{K}{n}v_j-(n-2)\beta u_j. \tag{2.8}$$

If we transvect (2.5) with $K_k{}^i$ and use (2.6), (2.7) and (2.8), then

$$K_{jt}K_k{}^t=\frac{K}{n}K_{jk}+(n-2)^2\beta^2(u_ju_k+v_jv_k)-\frac{n-2}{n}\beta K(u_jv_k+v_ju_k), \tag{2.9}$$

from which, using (2.5)

$$K_{jt}K_i{}^t=\frac{2}{n}KK_{ji}-\frac{K^2}{n^2}g_{ji}+(n-2)^2\beta^2(u_ju_i+v_jv_i). \tag{2.10}$$

LEMMA 2.1. *Let (M, g) be a special conformally flat space of dimension $n>2$ such that the scalar curvature K is constant and satisfies*

$$u^t\nabla_tu^h=0. \tag{2.11}$$

Then β is constant on (M, g).

Proof. Differentiating (2.5) covariantly and using (2.6), we have

$$-\frac{1}{n-2}\nabla_kK_{ji}=\beta_k(u_jv_i+v_ju_i)+\beta\{(\nabla_ku_j)v_i+(\nabla_kv_j)u_i+u_j\nabla_kv_i+v_j\nabla_ku_i\} \tag{2.12}$$

by virtue of K=constant.

Since K is constant, we see from (0.4) and (2.3) that

(2.13) $$\nabla_k K_{ji} - \nabla_j K_{ki} = 0.$$

Taking skew-symmetric parts of (2.12) with respect to k and j and using (2.13), we find

(2.14) $$\beta_k(u_j v_i + v_j u_i) - \beta_j(u_k v_i + v_k u_i) + \beta\{(\nabla_k u_j - \nabla_j u_k)v_i + (\nabla_k v_j - \nabla_j v_k)u_i + u_j \nabla_k v_i - u_k \nabla_j v_i + v_j \nabla_k u_i - v_k \nabla_j u_i\} = 0.$$

Transvecting (2.14) with $v^i u^j$ and using (2.11), $u^t v_t = 0$ and unit lengthies of u^h and v^h, we find

(2.15) $$\beta_k = \lambda u_k,$$

where we put $u^t \beta_t = \lambda$. From (2.15) we have

$$\nabla_k \beta_j = \lambda_k u_j + \lambda \nabla_k u_j,$$

from which, taking skew-symmetric parts,

$$\lambda_k u_j - \lambda_j u_k + \lambda(\nabla_k u_j - \nabla_j u_k) = 0.$$

Thus we have

(2.16) $$\lambda(\nabla_k u_j - \nabla_j u_k) = 0.$$

and

(2.17) $$\lambda_k = A u_k$$

for suitable function A.

We now restrict ourselves to the open set $M_0 \subset (M, g)$ where $\lambda(P) \neq 0$, then $\nabla_k u_j - \nabla_j u_k = 0$ on M_0 because of (2.16).

Consequently (2.14) becomes

(2.18) $$\lambda(u_k v_j - u_j v_k)u_i + \beta\{(\nabla_k v_j - \nabla_j v_k)u_i + u_j \nabla_k v_i - u_k \nabla_j v_i + v_j \nabla_k u_i - v_k \nabla_j u_i\} = 0$$

on M_0 because of (2.15). Transvecting (2.18) with u^i, we get

(2.19) $$\lambda(u_k v_j - v_k u_j) + \beta\{\nabla_k v_j - \nabla_j v_k + u_j u^t \nabla_k v_t - u_k u^t \nabla_j v_t\} = 0$$

on M_0, from which, transvecting $u^k v^j$ and using (2.11),

(2.20) $$\lambda + 2\beta v^s v^t \nabla_t u_s = 0.$$

on M_0.

Transvecting (2.18) with $v^j v^i$ and using (2.20), we find on M_0

(2.21) $$\beta v^t \nabla_j u_t = -\frac{1}{2}\lambda v_j.$$

Thus (2.19) becomes

(2.22) $$\beta(\nabla_k v_j - \nabla_j v_k) = -\frac{1}{2}\lambda(u_k v_j - u_j v_k)$$

and consequently (2.18) reduces to

(2.23) $$\frac{1}{2}\lambda(u_k v_j - u_j v_k)u_i + \beta(u_j \nabla_k v_i - u_k \nabla_j v_i + v_j \nabla_k u_i - v_k \nabla_j u_i) = 0$$

on M_0.

On the other hand, transvecting (2.12) with g^{ki} and taking account of (2.11), (2.15) and K=constant, we have

$$\lambda v_j+\beta\{v^t\nabla_t u_j+u^t\nabla_t v_j+(\nabla_t v^t)u_j+(\nabla_t u^t)v_j\}=0. \tag{2.24}$$

From (2.21) and (2.24) we easily see that

$$\beta u^t\nabla_t v_j=0 \tag{2.25}$$

on M_0.

Transvecting (2.23) with u^j and making use of (2.25), we have

$$\beta\nabla_k v_i=\frac{1}{2}\lambda v_k u_i \tag{2.26}$$

on M_0, from which, using (2.23),

$$\beta\nabla_k u_i=-\frac{1}{2}\lambda v_k v_i \tag{2.27}$$

on M_0.

Differentiating (2.26) covariantly along M_0, we obtain

$$\beta_k\nabla_j v_i+\beta\nabla_k\nabla_j v_i=\frac{1}{2}\lambda_k v_j u_i+\frac{1}{2}\lambda\{(\nabla_k v_j)u_i+v_j(\nabla_k u_i)\}, \tag{2.28}$$

or, substituting (2.15), (2.17), (2.26) and (2.27),

$$\beta^2\nabla_k\nabla_j v_i=\frac{1}{2}(A\beta-\lambda^2)u_k u_i v_j+\frac{1}{4}\lambda^2 v_k(u_j u_i-v_j v_i)$$

on M_0, from which, taking skew-symmetric parts with respect to k and j,

$$\beta^2(\nabla_k\nabla_j v_i-\nabla_j\nabla_k v_i)=\frac{1}{2}(A\beta-\frac{3}{2}\lambda^2)(u_k v_j-u_j v_k)u_i,$$

or, using Ricci identity

$$-\beta^2 K_{kjit}v^t=\frac{1}{2}(A\beta-\frac{3}{2}\lambda^2)(u_k v_j-u_j v_k)u_i \tag{2.29}$$

on M_0.

Transvecting (2.4) with v^h, we have

$$K_{kjit}v^t=2\alpha(v_j g_{ki}-v_k g_{ji})+\beta\{u_j g_{ki}-u_k g_{ji}+(v_j u_k-v_k u_j)v_i\}.$$

The last two equations imply that

$$-\beta^2[2\alpha(v_j g_{ki}-v_k g_{ji})+\beta\{u_j g_{ki}-u_k g_{ji}+(v_j u_k-v_k u_j)v_i\}] =\frac{1}{2}(A\beta-\frac{3}{2}\lambda^2)(u_k v_j-u_j v_k)u_i \tag{2.30}$$

on M_0. Transvecting this with $g^{ki}u^j$, we have $(n-2)\beta^3=0$, that is, $\beta=0$ on M^0.

Transvecting (2.30) with $u^k v^j u^i$ and using $\beta=0$, we have $\lambda=0$ on M_0. This contradicts our assumption, and consequently M_0 is a void set, i.e., β is a constant on (M,g). Thus Lemma 2.1 is proved.

LEMMA 2.2. *Under the same assumptions as those stated in Lemma* 2.1, *we have*

$$\nabla_j u_i=A_j v_i+2A_i v_j, \tag{2.31}$$

$$\nabla_j v_i=-A_j u_i-2A_i u_j, \tag{2.32}$$

and

$$A_j v^j=0 \tag{2.33}$$

provided that $\beta\neq 0$, *where*

$$A_j=v^t\nabla_j u_t. \tag{2.34}$$

Proof. Taking account of β=constant on (M, g), (2.12) and (2.14) can be respectively written as

$$-\frac{1}{n-2}\nabla_k K_{ji}=\beta\{(\nabla_k u_j)v_i+(\nabla_k v_j)u_i+u_j\nabla_k v_i+v_j\nabla_k u_i\}, \tag{2.35}$$

$$(\nabla_k u_j-\nabla_j u_k)v_i+(\nabla_k v_j-\nabla_j v_k)u_i+u_j\nabla_k v_i-u_k\nabla_j v_i+v_j\nabla_k u_i-v_k\nabla_j u_i=0 \tag{2.36}$$

by virtue of $\beta\neq 0$ on (M, g).

Transvecting (2.36) with v^i and u^i respectively and taking account of (2.34), we get

$$\nabla_k u_j-\nabla_j u_k=v_k A_j-v_j A_k, \tag{2.37}$$

$$\nabla_k v_j-\nabla_j v_k=u_j A_k-u_k A_j. \tag{2.38}$$

And further transvecting (2.38) with u^k and taking account of (2.11), we obtain

$$u^t\nabla_t v_j=-2A_j. \tag{2.39}$$

Similarly, by transvecting v^k to (2.37), we have

$$v^t\nabla_t u_j=2A_j-(v^t A_t)v_j. \tag{2.40}$$

Transvecting (2.36) with u^j and using (2.11), (2.17), we get

$$-A_k u_i-(u^t\nabla_t v_k)u_i+\nabla_k v_i-(u^t\nabla_t v_i)u_k=0,$$

or, using (2.39)

$$\nabla_k v_i=-A_k u_i-2A_i u_k. \tag{2.41}$$

Substituting (2.37) and (2.41) into (2.36), we find

$$(v_k A_j-v_j A_k)v_i+v_j\nabla_k u_i-v_k\nabla_j u_i=0,$$

from which, transvecting with v^j and using (2.40),

$$\nabla_k u_i=A_k v_i+2v_k A_i-2(A_t v^t)v_k v_i. \tag{2.42}$$

Substituting (2.41) and (2.42) into (2.35), we find

$$-\frac{1}{n-2}\nabla_k K_{ji}=2\beta\{-2(A_t v^t)v_k v_j v_i+A_k(v_j v_i-u_j u_i) \tag{2.43}$$

$$+A_j(v_k v_i-u_k u_i)+A_i(v_k v_j-u_k u_j)\}.$$

On the other hand, we have from (2.9),

$$K_{ji}K^{ji}=\frac{K^2}{n}+2(n-2)^2\beta^2, \tag{2.44}$$

from which, differentiating covariantly and using constancy of right hand side,

$$K_{ji}\nabla_k K^{ji}=0.$$

Transvecting (2.43) with K^{ji} and using the equation above, we obtain

$$0=\beta\{-2(A_t v^t)(K_{ji}v^j v^i)v_k-A_k(K_{ji}v^j v^i-K_{ji}u^j u^i)$$
$$+(K_{ji}v^i A^j)v_k-(K_{ji}u^i A^j)u_k+(K_{ji}v^j A^i)v_k-(K_{ji}u^j A^i)u_k\},$$

from which, using (2.7), (2.8) and (2.11), $(n-2)\beta^2 A_t v^t=0$ and consequently $A_t v^t=0$ because of $\beta\neq 0$. Thus (2.42) becomes (2.31). This completes the proof of Lemma 2.2.

LEMMA 2.3. *Under the same assumptions as those stated in Lemma* 2.1, *we have* $\beta=0$ *on* (M, g).

Proof. From Lemma 2.2, we see that (2.31) and (2.32) are valid when $\beta\neq 0$. Differentiating (2.31) covariantly, we find

$$\nabla_k \nabla_j u_i = (\nabla_k A_j) v_i + A_j (\nabla_k v_i) + 2(\nabla_k A_i) v_j + 2A_i (\nabla_k v_j),$$

from which, taking skew-symmetric parts with respect to indices k and j and using Ricci identity,

$$-K_{kjih} u^h = (\nabla_k A_j - \nabla_j A_k) v_i + A_j (\nabla_k v_i) - A_k (\nabla_j v_i) + 2(\nabla_k A_i) v_j - 2(\nabla_j A_i) v_k + 2A_i (\nabla_k v_j - \nabla_j v_k),$$

or, substituting (2.32)

$$-K_{kjih} u^h = (\nabla_k A_j - \nabla_j A_k) v_i + 2(v_j \nabla_k A_i - v_k \nabla_j A_i) + 4A_i (A_k u_j - A_j u_k),$$

from which, using (2.4)

(2.45) $$2\alpha (u_k g_{ji} - u_j g_{ki}) + \beta \{v_k g_{ji} + u_i (u_k v_j - u_j v_k) - v_j g_{ki}\} = (\nabla_k A_j - \nabla_j A_k) v_i + 2(v_j \nabla_k A_i - v_k \nabla_j A_i) + 4A_i (A_k u_j - A_j u_k).$$

On the other hand, we have from (2.31), (2.32) and (2.33)

(2.46) $$v^t \nabla_j A_t = -A^t \nabla_j v_t = 2(A_t A^t) u_j$$

and

(2.47) $$u^t \nabla_j A_t = -A^t \nabla_j u_t = -2(A_t A^t) v_j$$

by virtue of (2.11).

Transvecting (2.45) with v^i and using (2.33), (2.46) and (2.47), we have

(2.48) $$\nabla_k A_j - \nabla_j A_k = 2\alpha (u_k v_j - u_j v_k) - 4(A_t A^t)(u_k v_j - u_j v_k),$$

from which, transvecting $u^k v^j$ and using (2.46) and (2.47),

(2.49) $$\alpha = 4A_t A^t.$$

Since α is constant, by differentiating (2.46) covariantly, we have

(2.50) $$A^t \nabla_j A_t = 0.$$

Substituting (2.48) into (2.45) and taking account of (2.49), we find

$$2\alpha (u_k g_{ji} - u_j g_{ki}) + \beta \{v_k g_{ji} + (u_k v_j - u_j v_k) u_i - v_j g_{ki}\} = \alpha (u_k v_j - u_j v_k) v_i + 2(v_j \nabla_k A_i - v_k \nabla_j A_i) + 4A_i (A_k u_j - A_j u_k),$$

from which, transvecting $v^j A^i$ and using (2.11), (2.33) and (2.50), $\beta A_k = 0$ and consequently $A_k = 0$ because of $\beta \neq 0$. Thus (2.45) becomes

$$2\alpha (u_k g_{ji} - u_j g_{ki}) + \beta \{v_k g_{ji} + u_i (u_k v_j - u_j v_k) - v_j g_{ki}\} = 0.$$

If we transvect $v^k g^{ji}$ to this equation, then β must be zero. This contradicts the fact that $\beta \neq 0$ and hence $\beta = 0$ identically on (M, g). Thus Lemma 2.3 is proved.

Taking account of Lemma 2.3 and (2.4), we conclude the fact

THEOREM 2.4. *Let (M, g) be a special conformally flat space of dimension $n > 2$ with constant scalar curvature. If the associated vector field u^h of (M, g) is parallel along geodesics, then (M, g) is a space form.*

COROLLARY 2.5. *Let (M, g) be a conformally flat space of dimension $n > 2$ with mutually orthogonal unit vector fields u^h and v^h such that curvature operators associated with vector fields u^h and v^h respectively are of the forms*

(2.51) $$K_{kjih} u^k u^h = \mu (g_{ji} - u_j u_i),$$

(2.52) $$K_{kjih} v^k v^h = \nu (g_{ji} - v_j v_i)$$

μ and ν being constants. If the vector field u^h is parallel along geodesics, then (M, g)

is a space form.

Proof. Transvecting (2.2) with $u^k u^h$ and $v^k v^h$ respectively and using (2.51) and (2.52), we find

$$\mu(g_{ji}-u_j u_i)=-(L_{st}u^s u^t)g_{ji}+(L_{it}u^t)u_j-L_{ji}+(L_{jt}u^t)u_i \tag{2.53}$$

$$\nu(g_{ji}-v_j v_i)=-(L_{st}v^s v^t)g_{ji}+(L_{it}v^t)v_j-L_{ji}+(L_{jt}v^t)v_i. \tag{2.54}$$

Subtracting (2.54) from (2.53), we get

$$\begin{aligned}\{(\mu-\nu)+L(u,u)-L(v,v)\}g_{ji}-\mu u_j u_i+\nu v_j v_i \\ =(L_{it}u^t)u_j+(L_{jt}u^t)u_i-(L_{it}v^t)v_j-(L_{jt}v^t)v_i,\end{aligned} \tag{2.55}$$

where we have put

$$L_{st}u^s u^t=L(u,u), \qquad L_{st}v^s v^t=L(v,v).$$

Transvecting (2.55) with g^{ji}, $u^j u^i$ and $v^j v^i$ respectively, we obtain the relationships

$$\{\mu-\nu+L(u,u)-L(v,v)\}n-\mu+\nu=2L(u,u)-2L(v,v),$$

$$\mu-\nu+L(u,u)-L(v,v)-\mu=2L(u,u),$$

$$\mu-\nu+L(u,u)-L(v,v)+\nu=-2L(v,v).$$

The above three relations imply that

$$L(u,u)=L(v,v), \qquad \mu=\nu, \qquad L(u,u)=-\frac{1}{2}\mu. \tag{2.56}$$

Thus (2.55) reduces to

$$(L_{jt}u^t)u_i+(L_{it}u^t)u_j-(L_{jt}v^t)v_i-(L_{it}v^t)v_j=\mu(v_j v_i-u_j u_i).$$

Transvecting u^j and v^j to this equation respectively and using (2.56), we have

$$\begin{aligned}L_{it}u^t&=L(u,u)u_i+L(u,v)v_i,\\ L_{it}v^t&=L(u,v)u_i+L(u,u)v_i,\end{aligned} \tag{2.57}$$

where we have put $L(u,v)=L_{st}u^s v^t$.

Substituting (2.57) into (2.53) and taking account of the last the relation of (2.56), we obtain $L_{ji}=L(u,u)g_{ji}+L(u,v)(u_j v_i+v_j u_i)$.

Replacing α by $L(u,u)$ in (2.1), we have

$$K=-2n(n-1)L(u,u),$$

from which, using (2.56), K is constant. Thus all assumptions stated in Theorem 2.4 are valid, hence the conclusion of the theorem holds.

Finally we prove the following

THEOREM 2.6. *Let (M,g) be a compact special conformally flat space of dimension $n>2$ with non-negative constant scalar curvature K. Then (M,g) is a space form.*

Proof. Transvecting (2.10) with K^{ji} and using (2.7) and (2.8), we find

$$K_t{}^s K_s{}^r K_r{}^t=\frac{2}{n}KK_{ji}K^{ji}-\frac{1}{n^2}K^3+2(n-2)^2\beta^2\frac{1}{n}K,$$

or, using (2.44),

$$K_t{}^s K_s{}^r K_r{}^t=\frac{1}{n^2}K^3+\frac{6(n-2)^2}{n}\beta^2 K. \tag{2.58}$$

Substituting (2.44) and (2.58) into (1.1), we find

$$\frac{1}{2}\Delta(K_{ji}K^{ji})=\frac{n}{n-2}\left\{\frac{1}{n^2}K^3+\frac{6(n-2)^2}{n}\beta^2K\right\}-\frac{2n-1}{(n-1)(n-2)}K\left\{\frac{1}{n}K^2+2(n-2)^2\beta^2\right\}+\frac{1}{(n-1)(n-2)}K^3+(\nabla_jK_{ih})(\nabla^jK^{ih}),$$

or, equivalently

(2.59) $$\frac{1}{2}\Delta(K_{ji}K^{ji})=\frac{2(n-2)^2}{n-1}\beta^2K+(\nabla_jK_{ih})(\nabla^jK^{ih})\geqq 0$$

because K is non-negative. Since (M,g) is compact, using Green's theorem (cf. [9]), we have $\Delta(K_{ji}K^{ji})=0$ on (M,g) and consequently

(2.60) $$\nabla_kK_{ji}=0$$

and

$$\beta^2K=-2n(n-1)\beta^2\alpha=0$$

on (M,g) by virtue of (2.6). Since α is constant, $\alpha=0$ or $\beta=0$ on (M,g). Thus, if $\beta=0$, then from (2.4), (M,g) is a space form. But we can prove that $\alpha=0$ implies $\beta=0$. In fact, differentiating (2.5) covariantly and using $\alpha=0$ and (2.60), we have

(2.61) $$\beta_k(u_jv_i+v_ju_i)+\beta\{(\nabla_ku_j)v_i+(\nabla_kv_j)u_i+u_j\nabla_kv_i+v_j\nabla_ku_i\}=0,$$

from which, transvecting u^jv^i, $\beta_k=0$ and consequently β is constant on (M,g). Thus (2.61) becomes

$$\beta\{(\nabla_ku_j)v_i+(\nabla_kv_j)u_i+u_j\nabla_kv_i+v_j\nabla_ku_i\}=0.$$

If we assume $\beta\neq 0$ on (M,g), then

(2.62) $$(\nabla_ku_j)v_i+(\nabla_kv_j)u_i+u_j\nabla_kv_i+v_j\nabla_ku_i=0.$$

Transvecting (2.62) with u^j and v^j respectively, we have

(2.63) $$\nabla_kv_i=(v^t\nabla_ku_t)u_i,\qquad \nabla_ku_i=-(v^t\nabla_ku_t)v_i$$

by virtue of orthogonalities of u^h and v^h.

Substituting (2.63) into (2.62), we find

$$(v^t\nabla_ku_t)(u_ju_i-v_jv_i)=0.$$

from which, transvecting u^ju^i, $v^t\nabla_ku_t=0$ and so $\nabla_ju_i=0$, which implies $u^t\nabla_tu_j=0$. Therefore $\beta=0$ on (M,g) by virtue of Lemma 2.3.

Using the arguments developed above, we see that $\beta=0$ on (M,g) in any case whether α is zero or not. This completes the proof of Theorem 2.6.

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