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On semi-elastic spaces

by

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0. INTRODUCTION. We introduce a new class of topological spaces, semi-elastic spaces, which contains all elastic spaces and all semi-stratifiable spaces. Semi-stratifiable spaces are characterized by means of cushioned pair-net, from which it is easily shown that a semi-elastic space is a generalization of semi-stratifiable spaces. The main results of this note are:

(1) A necessary and sufficient condition for a topological space to be semi-stratifiable is that the space has a σ-cushioned pair-net.

(2) A semi-stratifiable space is semi-elastic.

(3) Every subspace of a semi-elastic space is semi-elastic.

1. SEMI-STRATIFIABLE SPACES. According to J.G.Ceder [2], a collection $\mathfrak{P}$ of ordered pairs $P=(P_1,P_2)$ of a space X is called a pair-base for X provided that P_1 is open for all $P\in\mathfrak{P}$ and that for every $x\in X$ and for every open U containing x, there exists a $P\in\mathfrak{P}$ such that $x\in P_1\subset P_2\subset U$.

A net-work (or net) in a space X is a collection $\mathfrak{N}$ of X such that given any $x\in X$ and any open subset U of X containing x, there exists an $N\in\mathfrak{N}$ such that $x\in N\in\mathfrak{N}$. A collection $\mathfrak{N}$ of ordered pairs $N=(N_1,N_2)$ of subsets of a space X is called a pair-net for X provided that for every $x\in X$ and for every open U containing x, there exists an $N=(N_1,N_2)\in\mathfrak{N}$ such that $x\in N_1\subset N_2\subset U$. Note that the first elements of members of a pair-net are not necessarily open.

Let $\mathfrak{Q}$ be a pair-base (or a pair-net) for a space X. $\mathfrak{Q}$ is said to be cushioned if for every subcollection $\mathfrak{Q}'$ of $\mathfrak{Q}$, we have

$$cl(\cup\{Q_1:Q\in\mathfrak{Q}'\})\subset\cup\{Q_2:Q\in\mathfrak{Q}'\}.$$

$\mathfrak{Q}$ is said to be σ-cushioned if it is a union of countably many cushioned subcollections.

G.D. Creede introduced the concept of semistratifiable spaces which is a generalization of that of stratifiable spaces [3]. The following definition is due to Creede.

Definition 1.1. A topological space X is semi-stratifiable space if, to each open set $U\subset X$, one can assign a sequence $\{U_n:n\in N\}$ of closed subsets of X such that

(a) $U=\cup\{U_n:n\in N\}$

(b) $U_n\subset V_n$ whenever $U\subset V$, where $\{V_n:n\in N\}$ is the sequence assigned to V.

Semi-stratifiable spaces can be characterized as the following

Theorem 1.2. A necessary and sufficient condition for a topological space to be semi-stratifiable is that the space has a σ-cushioned pair-net.

Proof. Let $U_n=S(n,U)$ for each $n\in N$ and for each open U (S is called a semi-stratification of X). Define $\mathfrak{N}_n=\{(S(n,U),U):U\in\tau\}$ for each $n\in N$, where τ is the topology of X. To show each $\mathfrak{N}_n$ is cushioned, let τ' be a subcollection of τ. Then $S(n,U)$ is contained in $S(n,\bigcup\{U:U\in\tau'\})$ for each $U\in\tau'$, and so

$$cl(\bigcup\{S(n,U):U\in\tau'\}\subset cl\ S(n,\bigcup\{U:U\in\tau'\})\subset S(n,\bigcup\{U:U\in\tau'\})\subset\bigcup\{U:U\in\tau'\}.$$

Conversely, suppose that there is a σ-cushioned pair-net $\mathfrak{N}=\bigcup\mathfrak{N}_n$ for X. For each $n\in N$ and $U\in\tau$, let

$$S(n,U)=cl(\bigcup\{N_1:N=(N_1,N_2)\in\mathfrak{N}_n \text{ and } N_2\subset U\}.$$

It is easily seen that S is a semi-stratification for X.

Semi-stratifiable spaces have many interesting properties which stratifiable spaces have:

Theorem 1.3. (Creede) A semistratifiable space is hereditarili semi-stratifiable.

Theorem 1.4. (Creede) The union of two closed semi-stratifiable subspaces is semi-stratifiable.

Of course, these properties can be proven by use of the concepts of pair-nets.

2. **SEMI-ELASTIC SPACES** To introduce the concept of elastic spaces, we need some definitions.

Definition 2.1. Let $\mathfrak{U}$ be a collection of subsets of a set X, and let $\mathfrak{R}$ be a relation on $\mathfrak{U}$ (i.e., $\mathfrak{R}\subset\mathfrak{U}\times\mathfrak{U}$). We shall often write $U\mathfrak{R}V$ instead of $(U,V)\in\mathfrak{R}$. The relation $\mathfrak{R}$ is said to be a framed relation on $\mathfrak{U}$ (or a framing of $\mathfrak{U}$) provided for every U, $V\in\mathfrak{U}$, if $U\cap V\neq\phi$, then $U\mathfrak{R}V$ or $V\mathfrak{R}U$. We say $\mathfrak{R}$ is a well-framed relation on $\mathfrak{U}$ provided $\mathfrak{R}$ is a framing of $\mathfrak{U}$ and for every $x\in X$, there is an $\mathfrak{R}$-smallest $U_x\in\mathfrak{U}$ containing x (i.e., if $x\in U$, $U\in\mathfrak{U}$, and $U\neq U_x$, then $(U,U_x)\notin\mathfrak{R}$).

Definition 2.2. A collection $\mathfrak{U}$ is said to be framed in a collection $\mathfrak{V}$ with frame map $f:\mathfrak{U}\to\mathfrak{V}$ provided there exists a framed relation R on $\mathfrak{U}$ such that for every subcollection $\mathfrak{U}'\subset\mathfrak{U}$ which has an $\mathfrak{R}$-upper bound (i.e. there exists $U\in\mathfrak{U}$ so that $U'\mathfrak{R}U$ for every $U'\in\mathfrak{U}'$) we have $cl(\bigcup\mathfrak{U}')\subset\bigcup f(\mathfrak{U}')$. If in addition R is a well-framed relation on $\mathfrak{U}$ we say that $\mathfrak{U}$ is well-framed in $\mathfrak{V}$ Finally, if $\mathfrak{U}$ is framed in $\mathfrak{V}$ and $\mathfrak{R}$ is also a transitive relation, then $\mathfrak{U}$ is called elastic in $\mathfrak{V}$, or an elastic refinement of $\mathfrak{V}$ when $\mathfrak{U}$ and $\mathfrak{V}$ are covers of X.

Theorem 2.3. Let X be a regular space. A necessary and sufficient condition that X be paracompact is that every open cover of X have an open elastic refinement([4]).

Definition 2.4. A pair-base $\mathfrak{P}$ for a space X is said to be an elastic base if there is a framing $\mathfrak{P}_1=\{p_1:P=(P_1,P_2)\in\mathfrak{P}\}$ such that $\mathfrak{P}_1$ is elastic in $\mathfrak{P}_2=\{P_1:P=(P_1,P_2)\in\mathfrak{P}\}$ with respect to the map $f(P_1)=P_2$. A T_1-space with an elastic base is called an elastic space.

This definition is due to Tamano and Vaughan[4]. We now generalize again the concept of elastic spaces.

Definition 2.5. A pair-net $\mathfrak{N}$ for a space X is said to be an elastic pair-net if there is a framing $\mathfrak{N}_1=\{N_1:N=(N_1,N_2)\in\mathfrak{N}\}$ such that $\mathfrak{N}_1$ is elastic in $\mathfrak{N}_2=\{N_2:N=(N_1,N_2)\in\mathfrak{N}\}$ with respect to the map $f(N_1)=N_2$. A space with an elastic pair-net is called a semi-elastic space.

Semi-elastic spaces have many properties in common with elastic spaces.

Theorem 2.6. Every subspaces of a semi-elastic space is semi-elastic.

Proof. Let $\mathfrak{N}$ be an elastic pair-net for a space X, and let Y be a subspace of X. Let $\mathfrak{W}$ denote the collection of all sets of the form $M=(M_1,M_2)$, where $N=(N_1,N_2)\in\mathfrak{N}$ and $M_i=N_i\cap Y$. Now it is easily verified that $\mathfrak{W}$ is an elastic pair-net for Y with the natural frame of $\mathfrak{W}$.

Theorem 2.7. (a) An elastic space is semi-elastic. (b) A semi-stratifiable space is semi-elastic.

Proof. (a) is clear. To show (b), let $\mathfrak{N}=\bigcup\mathfrak{N}_n$ be a σ-cushioned pair-net for a semi-stratifiable space X. Let $\leqq_n$ be an well-order of $\mathfrak{N}_n$ for each $n\in N$. Now define a frame on $\mathfrak{N}$ by

$$N\mathfrak{R}N' \text{ iff either } N\in\mathfrak{N}_n \text{ and } N'\in\mathfrak{N}_m \text{ with } n<m$$

or $N,N'\in\mathfrak{N}_n$ and $N\leqq_n N'$.

Now it is clear that $\mathfrak{N}$ is an elastic pair-net for X. This completes the proof.

Thus, we have the following diagram:

$$\begin{array}{ccc} \text{stratifiable} & \longrightarrow & \text{elastic} \\ \downarrow & & \downarrow \\ \text{semi-stratifiable} & \longrightarrow & \text{semi-elastic} \end{array}$$

As the following examples show, no arrows in the above diagram can be reversible.

Example 2.8. An elastic (hence, semi-elastic) space which is not semi-stratifiable. Let $X=[0,\Omega]$ be the set of ordinals less than or equal to the first uncountable ordinal. Let the topology on X be weakest topology stronger than the order topology for which every point is isolated except Ω. Construct an elastic base for X as follows. Let $U_\alpha=[\alpha,\Omega)$ for all $\alpha<\Omega$. and let $\mathfrak{P}'=\{(U,U):\alpha\in([0,\Omega))\}$ and order $\mathfrak{P}'$ by the usual order on the index set $([0,\Omega))$. Let $W_\alpha=\{\alpha\}$ for all $\alpha<\Omega$. and let $\mathfrak{P}''=\{(W_\alpha,W_\alpha):\alpha\in([0,\Omega))\}$ and order $\mathfrak{P}''$ by the usual order on the index set $[0,\Omega)$. Finally, set $\mathfrak{P}=\mathfrak{P}'\cup\mathfrak{P}''$ and order $\mathfrak{P}$ so that every element of $\mathfrak{P}'$ precedes every element of $\mathfrak{P}''$. Then $\mathfrak{P}$ is an elastic base for X, so X is an elastic space. Clearly X is not semi-stratifiable because the closed set $\{\Omega\}$ is not a G_δ in X (see Definition 1.1).

There are semi-stratifiable spaces which are not stratifiable. Such spaces are semi-elastic but not elastic.

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