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ON GENERALIZED INVERSE SPECTRUM

by

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We can extend the notion of an inverse spectrum to cover the case where more than one map between pairs of spaces is allowed. This extension is as follows:

Definition

Let A be a preordered set and $\{Y_\alpha | \alpha \in A\}$ be a family of spaces indexed by A. For each pair of elements α, β such that $\alpha < \beta$, let $\{\varphi^\mu_{\beta\alpha} | \mu \in M(\alpha, \beta)\}$ be a given nonempty family of continuous maps $\varphi^\mu_{\beta\alpha} : Y_\beta \to Y_\alpha$ which we will call connecting maps. The cardinality of $M(\alpha, \beta)$ need not be finite and may vary with (α, β). Assume that whenever $\alpha < \beta < \gamma$, then each $\varphi^\lambda_{\beta\alpha} \cdot \varphi^\mu_{\gamma\beta}$ is also a connecting map. Then $\{Y_\alpha; \varphi^\mu_{\beta\alpha}\}$ is called a generalized inverse spectrum over A. Each generalized inverse spectrum yields a limit space:

Definition

Let $\{Y_\alpha; \varphi^\mu_{\beta\alpha}\}$ be a generalized inverse spectrum over A. Form $\{Y_\alpha | \alpha \in A\}$, and for each α, let P_α be its projection onto the αth factor. The subspace $\{y \in \prod_\alpha Y_\alpha | \alpha, \beta \in A : [\alpha < \beta] \Rightarrow [P_\alpha(y) = \varphi^\mu_{\beta\alpha} \cdot P_\beta(y)$ for all $\mu \in M(\alpha,\beta)]\}$ is called the generalized inverse limit space of the generalized inverse spectrum and is denoted by Y_∞ or $\varprojlim Y_\alpha$.

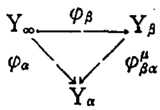
According to this definition, a point $y = \{y_\alpha\} \in \prod_\alpha Y_\alpha$ belongs to Y_∞ whenever $\alpha < \beta$ implies $y_\alpha = \varphi^\mu_{\beta\alpha}(y_\beta)$ for each $\mu \in M(\alpha, \beta)$. Since $\alpha < \alpha$ for each $\alpha \in A$, it follows that each coordinate y_α must actually belong to the subspace $A_\alpha = \{x \in Y_\alpha | \varphi^\lambda_{\alpha\alpha}(x) = x$ for each $\lambda \in M(\alpha,\alpha)\}$ of Y_α.

In fact, $\{A_\alpha; \varphi^\mu_{\alpha\beta} | A_\beta\}$ is itself a generalized inverse spectrum over A, since if $a_\beta \in A_\beta$, then for each pair $\alpha < \beta$ the formula $\varphi^\gamma_{\alpha\alpha} \cdot \varphi^\mu_{\beta\alpha}(a_\beta) = \varphi^\mu_{\beta\alpha}(a_\beta)$ shows $\varphi^\mu_{\beta\alpha}(a_\beta) \in A_\alpha$ for each $\lambda \in M(\alpha,\alpha)$ and $\mu \in M(\alpha,\beta)$ and it is easy to see that the two subspaces A_∞ and Y_∞ of $\prod_\alpha Y_\alpha$ are the same.

The elements of Y_∞ are also called threads; note that each threads has a unique representative in each Y_α, but that an element of Y_α may represent many threads. The restriction $P_\alpha | Y_\infty : Y_\infty \to Y_\alpha$ is denoted by φ_α and is called the canonical map of Y_∞ into Y_α. It is evidently continuous, and two threads x,y are the same if and only if $\varphi_\alpha(x)=\varphi_\alpha(y)$ for every $\alpha\in A$.

Now, we can obtain the following results.

THEOREM. (1) Whenever $\alpha<\beta$, the diagram



is commutative for each $\mu\in M(\alpha,\beta)$.

(2) If A is a directed set, then the sets $\{\varphi_\alpha^{-1}(U)\,|\,\text{all } \alpha, \text{ all open } U\subset Y_\alpha\}$ form a basis for Y_∞.

PROOF: (1) is obvious.

(2) Let $x\in V$, where V is open in Y_∞. Since Y_∞ is a subspace of $\prod_\alpha Y_\alpha$, there are finitely many open $U_{\alpha_i}\subset Y_{\alpha_i}$, $i=1,\ldots\ldots,n$ such that $x\in\langle U_{\alpha_1},\ldots\ldots,U_{\alpha_n}\rangle\cap Y_\infty\subset V$.

We are to show that for some suitable α and open $U\subset Y_\alpha$, $x\in\varphi_\alpha^{-1}(U)\subset\langle U_{\alpha_1},\ldots\ldots,U_{\alpha_n}\rangle\cap Y_\infty$. Because A is directed, we first choose α so that $\alpha_1,\ldots\ldots,\alpha_n<\alpha$, and then define $U=\bigcap_1^n\varphi^{\mu\;-1}_{\alpha\alpha_i}(U_{\alpha_i})$, which is open in Y_α. Now, according to (1), we have $\varphi_\alpha^{-1}(U)=\bigcap_1^n\varphi_\alpha^{-1}\varphi^{\mu\;-1}_{\alpha\alpha_i}(U_{\alpha_i})=\bigcap_1^n\varphi^{-1}_{\alpha_i}(U_{\alpha_i})$, so that a $y\in Y_\infty$ belongs to $\varphi_\alpha^{-1}(U)$ if and only if its α_ith coordinate lies in U_{α_i} for each $i=1,\ldots\ldots,n$; consequently, $y\in\varphi_\alpha^{-1}(U)\subset\langle U_{\alpha_1},\ldots\ldots,U_{\alpha_n}\rangle\cap Y_\infty$, as required.

THEOREM. Let $\{Y_\alpha;\ \varphi^\mu_{\beta\alpha}\}$ be an inverse spectrum over A.

(1) If each Y_α is Hausdorff, then Y_∞ is closed in $\prod_\alpha Y_\alpha$.

(2) If each Y_α is compact, then Y_∞ is compact.

PROOF: (1) Let $y=\{y_\alpha\}\in(\prod_\alpha Y_\alpha)-Y_\infty$. Then $\varphi^\mu_{\beta\alpha}(y_\beta)\neq y_\alpha$ for some pair $\alpha<\beta$ and for each $\mu\in M(\alpha,\beta)$.

Because Y_α is Hausdorff and $\varphi^\mu_{\beta\alpha}$ is continuous, we can find neighborhoods $U_\alpha(y_\alpha)$, $U_\beta(y_\beta)$ such that $U_\alpha\cap\varphi^\mu_{\beta\alpha}(U_\beta)=\phi$, and then is a neighborhood of y not meeting Y_∞.

(2) is an immediate consequence, since $\prod_\alpha Y_\alpha$ is compact.

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