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ON THE PRIME SPECTRUM OF A RING

by

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1. Introduction

Throughout this paper, we assume that every ring and ring homorphism will be understood to mean commutative ring with unit element and unitary homomorphism, respectively.

Let R be a ring and let X be the set of all prime ideals of R. Then X becomes a topological space endowed with the Zarski topology or the constructible topology, ([I]). Let X_z and X_c denote the set X endowed with the Zarski topology and the constructible topology, respectively.

In this note, we study homeomorphisms on X_z to X_c and X_z to the maximal spectrum Max (C(X)) for some conditions on the ring R, where C(X) is the ring of all real valued continuous function on the prime spectrum X_z

In the paragraph 3, we shall prove that X_z and Max(C(X)) are homeomorphic when R/N is regular where N is the nilradical of A. In the pragraph 4, we shall find some necessary and sufficient conditions that the Zarski topology and the constructible topology on X are the same.

2. Prelimiraries

Let R be a commutative ring with unit element. The ring R is called *reguler*, if every pincipal ideal of the ring is idempotent. We have the following properties for the regular ring.

Proposition 1. Let R be a regular ring, then every prime ideal of maximal.

Proof. Let P be a prime ideal and a an element of R such that $a \notin P$. Since R is regular, $(a)=(a^2)$, whence there exists an element x such that $a^2x=a$. Then $a(1-ax)=0 \in P$. Since $a \notin P$, $1-ax=p \in P$. That is $ax+p=1$ for some $x \in R$ and $p \in P$, Whence $(a,P)=(1)$. Therefore P is maximal.

Proposition 2. Let N be the nilradical of a ring R. If R/N is regular, then every prime ideal of R is maximal.

Proof. Let P be a prime ideal of R and let $a \notin P$. Since R/N is regular, $(\bar{a})=(\bar{a}^2)$, and hence there exists an $x \in R$ such that $\bar{a}=\bar{a}^2\bar{x}$. Therefore $a=xa^2+y$ for some $y \in N$. $a^n(1-xa)^n=y^n=0$ for some positive integer n. Hence $1-xa \in P$. Hence $(P,a)=1$, therefore P is maximal.

Proposition 3. Let R be a ring, A an ideal of R, M an R-module. Then $R/A \otimes_R M$ is isomorphic to M/AM.

Proof. The sequence $O \rightarrow A \xrightarrow{i} R \xrightarrow{j} R/A \rightarrow O$ is exact, where i is the inclusion and j is the natural homomorphism. Then $A\otimes M \xrightarrow{i\otimes 1} R\otimes M \xrightarrow{j\otimes 1} R/A\otimes M \to O$ is exact. Hence

$$R/A\otimes M = R\otimes M/\ker(j\otimes 1) = R\otimes M/\mathrm{im}(i\otimes 1).$$

Since $f: A\otimes_A M \to M$ detined by $f(a\otimes m) = am$ for $a\in R$ and $m\in M$ is an isomorphism and $f(\mathrm{im}(i\otimes 1) = AM$, $\mathrm{im}(i\otimes 1)\cong AM$. Therefore $R/A\otimes_A M \cong M/AM$.

Let S be a multiplicative subest of a ring R. The ring of fractions of R with respect to S is denoted by $S^{-1}R$. If $S = R - P$ for a prime ideal P of R, we write R_p for $S^{-1}R$, which is a local ring. If $S = \{f^n\}_{n\geq 0}$ for $f\in R$, we write R_f for $S^{-1}R$. There is a canonical homomorphism $\varphi: R \to S^{-1}R$ given by $\varphi(x) = x/1$. If A is an ideal of R, we define $A^e = AS^{-1}R = \{t(a/1) \mid a\in A,\ t\in S^{-1}R\}$, which is called the *extension* of A in $S^{-1}R$. On the otherhand if B is an ideal of $S^{-1}R$, we define $B^C = \varphi^{-1}(B)$, which is called the *contraction* of B to R. If $f: R \to R'$ is a ring homomorphism and S is a multiplicative subset of R, then $f(S)$ is multiplicative subest of R'. We write $S^{-1}R'$ for $f(S)^{-1}R$. If $S = R - P$, for a prime ideal P of R, we write R'_p for $S^{-1}R'$ and AR'_p for $f(A)R'_p$.

There exists a bijection between the set of prime ideals of R whose intersection with the multiplicative subset S of R is empty and the set of prime ideals of $S^{-1}R$, ([2]).

Let M be an R module and let S be a multiplicative subset of R then we denote the $S^{-1}R$-module of fractions as $S^{-1}M$. We M_p for $S^{-1}M$ when $S = R - P$ for a prime ideal P of R, and M_f when $S = \{f^n\}_{n\geq 0}$ for $f\in R$.

Proposition 4. Let M be an R-module. Then the $S^{-1}R$-module $S^{-1}M$ and $S^{-1}R\otimes_R M$ are isomorphic: i.e. there exists a unique isomorphism $f: S^{-1}R\otimes_R M \to S^{-1}M$ for which

$$f(a/s)\otimes m) = am/s \text{ for all } a\in A,\ m\in M,\ s\in S \qquad \text{(i)}$$

Proof. The mapping $S^{-1}R\times M \to S^{-1}M$ defined by $(a/s, m) \mapsto am/s$ is R-bilinear, and therefore by the universal property of the tensor product induces an R-homorphism $f: S^{-1}R\otimes_R M \to S^{-1}M$ satisfying (i). Clearly f is surjective, and is uniquely defined by (i).

Let $\sum(a_i/s_i)\otimes m_i$ be any element of $S^{-1}R\otimes M$. If $s = \Pi s_i \in S$, $t_i = \Pi_{i\neq j} s_j$ we have

$$\sum_i (a_i/s_i)\otimes m_i = \sum_i (a_i t_i/s)\otimes m_i = \sum_i (1/s)\otimes a_i t_i m_i$$
$$= (1/s)\otimes \sum_i a_i t_i m_i,$$

so that every element of $S^{-1}R\otimes M$ is of the form $(1/s)\otimes m$. Suppose that $f((1/s)\otimes m) = 0$. Then $m/s = 0$, hence $tm = 0$ for some $t\in S$, and therefore $(1/s)\otimes m = (t/st)\otimes m = (1/st)\otimes tm = (1/st)\otimes 0 = 0$. Hence f is injective and therefore an isomorphism.

Remark 1. Let I be a directed set and let $M = (m_i, \mu_{ij})$ be a direct system of R-modules over the directed set I. Then we have the direct limit of the direct system **M** consisting the module M and the family of homomorphisms $\mu_i: M_i \to M$, which we denote $M = \varinjlim M_i$. For this direct limit, we have the following properties. Every element of M can be written in the form $\mu_i(x_i)$ for some $i\in I$ and $x_i\in M_i$. If $\mu_i(x_i) = 0$, then there exists $j\geqq i$ such that $\mu_{ij}(x_i) = 0$ in M_j.

Remark 2. Keeping the same notation as in Remark 1, let N be any R-module. Then $(M_i\otimes N, \mu_{ij}\otimes 1)$ is a direct system; let $P = \varinjlim(M_i\otimes N)$ be its direct limit. For each $i\in I$, we have a homomorphism $\mu_i\otimes 1: M_i\otimes N \to M_i\otimes N$. Then we have the unique homomorphism $\psi: P \to M\otimes N$ by the definition of the direct limit. We can easily prove that the homomorphism ψ is an isomorphism, so that $\varinjlim(M_i\otimes N) \cong (\lim M_i)\otimes N$, ([1]).

Remark 3. Let $\{R_i\}_{i\in I}$ be a family of rings indexed by a directed set I, and for each pair $i\leqq j$ in I, let α_{ij}: $R_i\to R_j$ be a ring homomorphism such that α_{ii} is the identity mapping of R_i for all $i\in I$, and $\alpha_{ik}=\alpha_{jk}\alpha_{ij}$ whenever $i\leqq j\leqq k$. Regarding each R as a **Z**-module where **Z** is the ring of integers, we can then form the direct limit $R=\varinjlim R_i$. Then R inherits a ring structure from the R_i so that the mappings $R_i\to R$ are ring homomorphisms. The ring R is the direct limit of the system(R_i, α_{ij}). For this direct limit we can easily prove the following results by Remark 1. If $R=0$ then $R_i=0$ for some $i\in I$.

Remark 4. Let $\{R_i\}$ $i\in I$ be a family of Λ-algebras. For each finite subset J of I let R_J denote the tensor product (overΛ) of the R_i for $i\in J$. If J' is another finite subset of I and $J\subset J'$ there is a canonical Λ-algebra homomorphism $R_J\to R_{J'}$. Let R denote the direct limit of the rings R_J as J runs through all finite subsets of I. The ring R has a natural Λ-algebra structure for which the homomorphisms $R_J\to R_{J'}$ are Λ-algebra homomorphisms. The Λ-algebra R is the tensor product of the family $\{R_i\}_{i\in I}$.

3. Prime spectrum of a ring

For each subset E of a ring R, let V(E) denote the set of all prime ideals of R which contain E. Then the following are easily proved.

(1) If E is a subset of R, A is the ideal generated by E, and r(A) is the radical of A, then $V(E)=V(A)=V(r(A))$.

(2) If $\{E_i\}_{i\in I}$ is any family of subsets R, then $V(\bigcup_{i\in I}E_i)\bigcap_{i\in I}V(E_i)$.

(3) If A,B are ideals of R, then $V(A)\bigcup V(B)=V(A\bigcap B)=V(AB)$.

Thease results show that the sets V(E) satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zarski topology.* The topological space X is called the *prime spectrum* of R, and is written Spec(R). For each $f\in R$, let X_f denote the complement of V(f) in X=Spec(R). The sets X_f are open.

Proposition 5. The sets X_f from a basis of open sets for the Zarski topology.

Proof. Let G be an open set and $x\in G$, then $X-G=V(E)$ for some subset of R, and $x\notin V(E)$. Hence there exists an element f of E such that $f\notin x$. Then $x\in X_f$. Hence $X_f=X-V(f)\subset X-V(E)=G$. Therefore $\{X_f\}$ form a basis of the topology.

Proposition 6. For any $f\in R$, X_f is quasi-compact.

Proof. Let $\{X_{f_i}|i\in I\}$ be a basic open covering of X_f. Then $X_f\subset\bigcup_{i\in I}X_{f_i}$, whence $V(f)\supset\bigcap_{i\in I}V(X_{f_i})=V(\{f_i|i\in I\})$. Hence $r(\{f_i|i\in I\})\supset r(f)$, since the radical of an ideal is the intersection of all prime ideals containing the ideal. Since $f\in r(f)$, there exists a positive ingeger n such that $f^n\in(\{f_i|\in I\})$, whence $f^n=\sum_{i=1}^{k} x_if_i$ for some positive integer k and $x_i\in A$. Since$(f^n)=(\sum x_if_i)\subset\sum(f_i)$, $V(f)=V(f^n)=V(\sum x_if_i)\supset V(\sum(f_i))=V(\bigcup(f_i))=\bigcap V(f_i)$. Hence $X_f\subset\bigcup X_{f_i}$. Therefore X_f is quasi-compact.

Proposition 7. For a ring R the following are equivalent:

(1) Every prime ideal of R is maximal.

(2) X=Spec(R) is a T_1-space.

Proof. (1)$\Rightarrow$(2): Let$\{x\}$ be a one point subset of X. Since x is a maximal ideal of R,$V(x)=x$, hence $\{x\}$ is closed, Therefore X is a T_1-space.

(2)⇒(1); Let x be a prime ideal of R, then $\{x\}$ is closed subset of X. Hence $\{x\} = \overline{\{x\}} = V(x)$. Therefore x is maximal.

Proposition 8. Let N be the nilradical of a ring R. If R/N is regular, X=Spec(A) is Hausdorff.

Proof. Let x,y be distinct two elements of X, then there exists an element f such that $f \in y-x$ or $f \in x-y$. Suppose that $f \in y-x$. Since R/N is regular, $(\bar{f}) = (\bar{f}^2)$, whence there exists an element o of R such that $\bar{f} = \bar{a}\bar{f}^2$, where $a \notin x$. Let $af = e$, then $e \notin N$, $e \in y$ and $\bar{e}$ is an idempotent of R/N and $(\bar{e}) = (\bar{f})$. Let $1-e=g$, then $g \notin y$ and $eg \in N$, because $\bar{g}$ is also an idempotent of R/N. Then $y \in X_g$ and $X_f \cap X_g = X_{fg} = \phi$. Hence X_f and X_g are disjoint neighbourhood of x and y respectively. Therefore X is Hausdorff.

Let φ: R→S be a ring homomorphism. Let X=Spec(R) and Y=Spec(S). If $Q \in Y$, then $\varphi^{-1}(Q)$ is a prime ideal of R, i.e. a point of X. Hence φ induces a mapping φ^*: Y→X.

Proposition 9. The mapping φ^*: Y→X is continuous.

Proof. Let f be any element of R, then it is clear that $\varphi^{*-1}(X_f) = Y_{\varphi(f)}$ by the following results:

$$Q \in \varphi^{*-1}(X_f) \Longleftrightarrow \varphi^*(Q) \in X_f \Longleftrightarrow f \notin \varphi^*(Q) \Longleftrightarrow f \notin \varphi^{-1}(Q) \Longleftrightarrow \varphi(f) \notin Q \Longleftrightarrow Q \in X_{\varphi(f)}.$$

Therefore φ^* is continuous, since the inverse image of every basic open set X_f is open in Y.

Let R be a ring. The subspace of Spec(R) consisting of maximal ideals of R, with the induced topology, is called the *maximal spectrum* of R and is denoted by Max(R).

Let X=Spec(R) and let C(X) denote the ring of all real-valued continuous functions on X. We can add and multiply functions in C(X) by adding and multiplying their values. Then C(X) becomes a commutative ring with unit. We have the following property concerning C(X).

Theorem 1. Let X=Max(C(X)) be the maximal spectrum of C(X). If R/N is regular, then we have an homeouorphism $\mu . X \to \tilde{X}$, where N is the nilradical of R.

Proof. Since R/N is regular, X is Hausdorff compact space by Proposition 6 and 8. For each $x \in X$, let M_x be the set of all $f \in C(X)$ such that $f(x)=0$. The ideal M_x is maximal, because it is the kernel of the surjective homomorphism $C(X) \to \mathbf{R}$ which takes to f(x), where **R** is the field of real numbers. Then we can define a mapping $\mu : X \to \tilde{X}$, namly $x | \to M_x$.

Let M be any maximal ideal of C(X), and let V=V(M) be the set of common zeros of the functions in M: that is,

$$V = \{x \mid x \in X,\ f(x)=0 \text{ for all } f \in M\}$$

Suppose that V is empty. Then for each $x \in X$ there exists $f_x \in M$ such that $f_x(x) \neq 0$. Since f_x is continuous, there is an open neighbourhood U_x of in X on which f_x does not vanish. By compactness a finite number of neighbourhoods, say $U_{x_1}, U_{x_2}, \ldots\ldots U_{x_n}$, cover X. Let $f = f^2_{x_1} + f^2_{x_2} + \ldots\ldots + f^2_{x_n}$. Then f does not vanish at any point of X, hence is a unit in C(X). But this contradicts $f \in M$, hence V is not empty.

Let x be a point of V. Then $M \subset M_x$ hence $M = M_x$ because M is maximal. Hence μ is surjective.

Since X is hausdorff and compact, X is normal. Hence by Urysohn's lemma the continuous functions separate the points of X. Hence $x \neq y \to M_x \neq M_y$, and therefore μ is injective.

Let $f \in C(X)$, let $U_f = \{x \mid x \in X,\ f(x)=0\}$ and let $\tilde{U}_f = \{M \mid M \in \tilde{X},\ f \in M\}$. Then $\mu(U_f) = \tilde{U}_f$ and the open sets U_f(resp. $\tilde{U}_f$) form a basis of the topology of X(resp. $\tilde{X}$) and therefore μ is homeomorphism.

4. Constructible topology on Spec(R).

We construct another topology on Spec(R) which we call the constructible topology on Spec (R), and then we compare the topology with the Zarski topology in this paragraph.

Theorem 2. Let $f: R \to R'$ be a ring homomorphism, let P be a prime ideal of R and let $S = R - P$. Then the subspace $f^{*-1}(P)$ of Y (=Spec(R')) is homeomorphic to $\mathrm{Spec}(R'_P/PR'_P) \cong \mathrm{Spec}\ (k(p) \otimes_R R')$, where f^* is a mapping on Y to X(=Spec(R)) defined by $f^*(Q) = f^{-1}(Q)$, and $k(p)$ is the residue field of the local ring R_P.

Proof. For any $Q \in f^{*+1}(P)$, $S^{-1}Q$ is the prime ideal of R'_P and $PR'_P \subset S^{-1}Q$. Hence $S^{-1}Q/PR'_P$ is a prime ideal of R'_P/PR'_P. Let us define the mapping $\psi: f^{*+1}(P) \to \mathrm{Sped}(R'_P/PR'_P)$ by $\psi(Q) = S^{+1}Q/PR'_P$ for any $Q \in f^{*-1}(P)$. We will accomplish the proof by a series of reduction.

(a) Let $Q' \in \mathrm{Spec}(R'_P/PR'_P)$ be any prime ideal of R'_P/PR'_P and let $Q'' = g^{-1}(Q')$ for the natural homomorphism $g: R'_P \to R'_P/PR'_P$. Then Q'' is a prime ideal of R'_P and there exists a prime ideal Q of R' such that $S^{-1}Q = Q''$, and then $S^{-1}Q/PR'_P = Q'$ and $f^{-1}(Q) \cap S = \phi$. Let $f^{-1}(Q) = P'$, then $P' \cap S = \phi$, whence $P' \subset P$. On the otherhand, since $Q'' \supset PR'_P$ and $Q = \varphi^{-1}(Q'') \supset \varphi^{-1}(PR'_P) \supset f(P)$ for the canonical homomorphism $\varphi: R' \to R'_P$ by defined $\varphi(x) = x/1$, $P' \supset P$. Therefore $P' = P$, whence $Q \in f^{*-1}(P)$, and $\psi(Q) = S^{-1}Q/PR' = Q'$. Therefore ψ is surjective.

(b) If $\psi(Q_1) = \psi(Q_2)$ for any Q_1, Q_2 in $f^{*-1}(P)$, then $S^{-1}Q_1/PR'_P = S^{-1}Q_2/PR'_P$, whence $S^{-1}Q_1 = S^{-1}Q_2$, $Q^1 = Q^2$. Therefore ψ is injective.

(c) Let C be any closed subset of $f^{*-1}(P)$, then there exists an ideal B of R' such that $C = V(B) \cap f^{*-1}(P)$. We will prove $(V(B) \cap f^{*-1}(P)) = V(S^{-1}B/PR'_P)$. If $Q' \in \psi(V(B) \cap f^{*-1}(P))$, $Q' = \psi(Q)$ for some prime ideal Q of R' such that $B \subset Q$ and $f^{-1}(Q) = P$. Then $S^{-1}B \subset S^{-1}Q$ and $Q' = \psi(Q) = S^{-1}Q/PR'_P \supset S^{-1}B/PR'_P$, whence $Q' \in V(S^{-1}B/PR'_P)$. Hence $\psi(V(B) \cap f^{*-1}(P)) \subset V(S^{-1}B/PR'_P)$. Conversely, if $Q' \in V(S^{-1}B/PR_P')$, $Q' \supset S^{-1}B/PR'_P$. Let $Q'' = g^{-1}(Q')$ for the natural homomorphism $g: R'_P \to R'_P/PR'_p$, then Q'' is a prime ideal of R'_P and $Q' = Q''/PR'_P \supset S^{-1}/PR'_P$, whence $Q'' \supset S^{-1}B$. Let $Q' = \varphi^{-1}(Q'')$ for the canonical homomorphism $\varphi: R' \to R'_P$ defined by $\varphi(x) = x/1$. Then $Q = \varphi^{-1}(Q'') \supset \varphi^{-1}(PR'_P) \supset f(P)$, whence $f^{-1}(Q) \supset P$. On the otherhand, since $Q \cap f(S) = \phi$, $f^{-1}(Q) \subset P$. Hence $f^{-1}(Q) = P$.Since $Q'' \supset S^{-1}B$ and $Q = \varphi^{-1}(Q'') \supset \varphi^{-1}(S^{-1}B) \supset B$, $Q \in V(B)$. Therefore $Q \in V(B) \cap f^{*-1}(P)$ and $\psi(Q) = Q' \in \psi(V(B) \cap f^{*-1}(P))$. Therefore $V(S^{-1}B/PR') \subset \psi(V(B) \cap f^{*-1}(P))$, whence $\psi(V(B) \cap f^{*-1}(P)) = V(S^{-1}B/PR'_P)$. Therefore ψ is the closed mapping.

(d) Let D be a closed subset of $\mathrm{Spec}(R'_P/PR'_P)$, then there exists an ideal C of R'_P/PR'_P such that $D = V(C)$. Let $B' = g^{-1}(C)$ for the natural homomorphism $g: R' \to R'_P/PR'_P$, then $C = B'/PR'_P$. Let $B = \varphi^{-1}(B')$ for the canonical homomorphism $\varphi: R' \to R'_P$ defined by $\varphi(x) = x/1$, then $B' = S^{-1}B$, whence $C = S^{-1}B/PR'_P$. Then $\psi(V(B) \cap f^{*-1}(P)) = V(C) = D$. The set $V(B) \cap f^{*-1}(P)$ is a closed subset of $f^{*-1}(P)$. Hence ψ is continuous mapping. Therefore ψ is a homeomorphism.

(e) Since $k(P) = R_P/S^{-1}P$, $k(P) \otimes_{R_P} R'_P = R_P/S^{-1}P \otimes_{R_P} R'_P \cong R'_P/S^{-1}PR'_P$ by Proposition 3. Since

$S^{-1}PR'_P = PR'_P$ and $R'_P \cong R_P \otimes_R R'$, by Propositions 4, $R'_P/PR'_P \cong k(P) \otimes_{R_P} R'_P \cong k(P) \otimes_{R_P} (R_P \otimes_R R') \cong (k(P) \otimes_{R_P} R_P) \otimes_R R' \cong k(p) \otimes_R R'$. This completes the proof of the theorem.

Corallary. Let $f: R \to R''$, $g: R \to R''$ be ring homomorphisms and let $h: R \to R' \otimes_R R''$ be defined by $h(x) = f(x) \otimes g(x)$. Let X, Y, Z, T be the prime spectra of R, R', R'', $R' \otimes_R R''$ respectively. Then $h^*(T) = f^*(Y) \cap g^*(Z)$.

Proof. Let $P \in X$, and let $k = k(P)$ be the residue field at P. By the above theorem, the fiber $h^{*-1}(p)$ is the spectrum of $(R' \otimes_R R'') \otimes_R k \cong (R' \otimes_R k) \otimes_k (R'' \otimes_R k)$. Hence $P \in h^*(T)$ $(R' \otimes_R k) \otimes_k (R'' \otimes_R k) \neq 0 \Longleftrightarrow R' \otimes_R k \neq 0$ and $R'' \otimes_R k \neq 0 \Longleftrightarrow P \in f^*(Y) \cap g^*(Z)$. Therefore $h^*(T) = f^*(Y) \cap g^*(Z)$.

Proposition 10. Let $(R_\alpha, g_{\alpha\beta})$ be a direct system of rings and R' the direct limit. For each α, let $f_\alpha: R \to R_\alpha$ be a ring homomorphism such that $g_{\alpha\beta} f_\alpha = f_\beta$ whenever $\alpha \leqq \beta$. If $f: R \to R'$ is the ring homomorphism induced by f_α, then $f^*(\mathrm{Spec}(R')) = \cap_\alpha f^*_\alpha(\mathrm{Spec}(R_\alpha))$.

Proof. Let $P \in \mathrm{Spec}(R')$. Then $f^{*-1}(P)$ is the spectrum of $R' \otimes_R k(P)$ by Theorem 2, and $R' \otimes_R k(P) \cong \varinjlim (R_\alpha \otimes_R k(P))$, since tensor products commute with direct limits. By Remark 3 it follows that $f^{*-1}(P) = \phi$ if and only if $R_\alpha \otimes k(P) = 0$ for some α, *i, e.*, if and only if $f^{*-1}(P) = \phi$. Hence $f^{*-1}(P) \neq \phi$ if and only if $f_\alpha^{*-1}(P) = \phi$ for all α and hence $p \in f^*(\mathrm{Spec}(R'))$ if and only if $P \in \cap_\alpha f_\alpha^*(\mathrm{Spec}(R_\alpha))$. Therefore $f^*(\mathrm{Spec}(R') = \cap_\alpha f^*_\alpha(\mathrm{Spec}(R_\alpha))$.

Proposition 11. Let $f_\alpha: R \to R_\alpha$ be any family of R-algebras and let $f: R \to R'$ be their tensor product over R. Then $f^*(\mathrm{Spec}(R')) = \cap_\alpha f^*_\alpha(\mathrm{Spec}(R_\alpha))$.

Proof. Let I be the index set of α. For each finite subset J of I, let R_J denote the tensor product over R of the R_α for $\alpha \in J$. Then (R_J, g_{JK}) is a direct system of R-algebras, where g_{JK}: $R_J \to R_K$ is the canonical R-algebra homomorphism for $J \subset K$. Then the tensor product R' of the family $\{R_\alpha\}_{\alpha \in I}$ is $R' = \varinjlim R_J$ for which the homomorphisms $f_J: R_J \to R'$ are R-algebra homomorphisms. Hence $f^*(\mathrm{Spec}(R')) = \cap_J f^*_J(\mathrm{Spec}(R_J))$. But $f^*_J(\mathrm{Spec}(R_J) = \cap_{\alpha \in J} f_\alpha^*(\mathrm{Spec}(R_\alpha))$ by the above corollary of Theorem 2. Therefore $f^*(\mathrm{Spec}(R')) = \cap_\alpha f^*_\alpha(\mathrm{Spec}(R_\alpha))$.

Proposition 12. Let $f_1: R \to R_1$ and $f_2: R \to R_2$ be any R-algebra homomorphisms and let $R' = R_1 \times R_2$. If we define $f: R \to R'$ by $f(x) = (f_1(x), f_2(x))$, then $f^*(\mathrm{Spec}(R')) = f^*_1(\mathrm{Spec}(R_1)) \cap f^*_2(\mathrm{Spec}(R_2))$.

Proof. Let X_1 and X_2 be the sets of all prime ideals of R' containing $\{0\} \times R_2$ and $R_1 \times \{0\}$ respectively, then $\mathrm{Spec}(R') = X_1 \cup X_2$, whence $f^*(\mathrm{Spec}(R')) = f^*(X_1 \cup X_2) = f^*(X_1) \cup f^*(X_2)$. Since $f^*(X_1) = f_2^*(\mathrm{Spec}(R_1))$ and $f^*(X_2) = f^*_2(\mathrm{Spec}(P_2))$, $f^*(\mathrm{Spec}(R')) = f^*_1(\mathrm{Spec}(R_1)) \cup f_2^*(\mathrm{Spec}(R_2))$.

By Propostion 11 and 12, the subsets of $X = \mathrm{Spec}(R)$ of the form $f^*(\mathrm{Spec}(R'))$, where $f: R \to R'$ is a ring homomorphism, satisfy the axioms for closed sets in a topological space. The associated topology is the ***constructible topology*** on X.

Proposition 13. The constructible topology is finer than the Zarski topology.

Proof. Let F be any closed subset of X in the Zarski topology, then there exists an ideal A of R such that $F = V(A)$. Since $V(A) = \varphi^*(\mathrm{Spec}(R/A))$ for the natural homomorphism φ: $R \to R/A$, F is a closed subset of X in the constructible topology. Therefore the constructible topology is finear than the Zarski topology.

Proposition 14. Let X_C denote the set $X = \mathrm{Spec}(R)$ of a ring R endowed with the constructible topology. Then X_C is quasi-compact.

Proof. Let $\{F_\alpha\}_{\alpha\in p}$ be a family of closed subsets of X_C such that $\bigcap_\alpha F_\alpha=\phi$. There exist the ring homomorphisms $f_\alpha:R\to R_\alpha$ such that $F_\alpha=f^*_\alpha(\mathrm{Spec}(R_\alpha))$. Let R' be the tensor product of the family of R-algebras $f_\alpha:R\to R_\alpha$ over R. Then

$$\mathrm{Spec}(R')=\bigcap_\alpha f^*_\alpha(\mathrm{Spec}(R_\alpha))=\bigcap_\alpha F_\alpha=\phi$$

Hence $R'=0$. Therefore there is finite subset $\{\alpha_1,\alpha_2,\ldots\ldots\alpha_n\}$ of I such tha $R_{\alpha 1}\otimes R_{\alpha 2}\otimes\ldots\ldots\otimes R_{\alpha_n}=0$. Then $\mathrm{Spec}(R_{\alpha_1}\otimes\ldots\ldots\otimes R_{\alpha_n})=\bigcap_{i=1}^{n}(\mathrm{Spec}(R_{\alpha i}))=\bigcap_{i=1}^{n} F_{\alpha i}=\phi$. Therefore X_C is quasi-compact.

Proposition 15. For each $g\in R$, X_g is both open and closed in the constructible topology.

Proof. Since the Zarski topology Z is weaker than the constructible topology C and $X_g\in Z$, $X_g\in C$. Hence X_g is open in C. Let $S=\{g^n|n\text{:non-negative integers}\}$, and let $\varphi:R\to S^{-1}R$ be the canonical ring homomorphism defined by $\varphi(a)=a/1$. Then $X_g=\varphi^*(\mathrm{Spec}(S^{-1}R))$. Therefore X_g is closed in C.

Proposition 16. X_C is the Hausdorff space.

Proof. Let P, Q be any distinct elements of X_C, then there exist an element such that $g\in Q-P$ or $g\in P-Q$. Suppose $g\in Q-P$. Let $1-g=f$, then $P\in X_g$ and $Q\in X_f$, whence $Q\in X_f-X_g$. Since X_g, X_f-X_g are open and $X_g\cap(X_f-X_g)=\phi$, these two sets are disjoint neighbourhood of P and Q respectively. Therefcre X_C is Hausdorff.

Lemma. Let T_1 and T_2 are two topologies on a set X such that $T_1\subset T_2$. If (X,T_1) is Hausdorff and (X,T_2) is quasi-compact, then $T_1=T_2$.

Proof. Since (X,T_2) is quasi-compact and $T_1\subset T_2$, (X,T_1) is also quasi-compact. Let F be any closed set in T_2, and let $\{G_\alpha\}_{\alpha\in I}$ be any open covering of F in (X,T_1). Then the family is also open covering in (X, T_2). Hence there exists a finite subcovering $\{G_{\alpha_1},G_{\alpha 2},\ldots\ldots,G_{\alpha n}\}$ of $\{G_\alpha\}_{\alpha\in I}$. Therefore F is quasi-compact in (X, T_1). Since(X,T_1) is Hausdorff,F is closed in T_1. Hence $T_1=T_2$.

Theorem 3. Let X be the prime spectrum of a ring R, let Z and C be the Zarski topology and the constructible topogy of X respectively, and let N be the niladical of R. Then the following are equivalent:

(1) R/N is regular

(2) X_Z is Hauosdorff.

(3) Z=C

(4) For each $f\in R$, X_f is open and closed in X_Z

Proof. (1)⇒(2); It is clear by Proposition 8.

(2)⇒(3); It is clear by Proposition 14 and Lemma.

(3)⇒(4); It is clear by Proposition 15.

(4)⇒(1): we can prove that the space X_Z is Hausdorff in the same way as the proof of proposition 15. Then we know that every prime ideal of R is maximal by proposition 7. Let f be any element of R, and let P be any element of X_f. Then $f\notin P$, and there exists $a\in R$ and $g\in P$ such that $g+af=1$, since P is maximal. Since $g=1-af\in P$, $P\in V(1-af)$. Since $V(1-af)$ is an open and closed subset of X_Z, the family $\{V(1-af)|a\in R\}$ is an open covering of X_f. Since X_f is quasi-compact, there exists a finite subset $\{a_1, a_2,\ldots\ldots,a_n\}$ of R such that $X_f\subset$

$\subset_{i=1}^{n} V(1-a_i f)$. Hence $X_f \cap (\cap_{i=1}^{n} X_{1-a_i f}) = \phi$. Since $\cap_{j=1}^{n} X_{1-a_i f} = X_{\Pi_{i=1}^{n}(1-a_i f)}$ and $\Pi_{i=1}^{n}(1-a_i f) = 1-af$ for some $a \in R$, $X_f \cap X_{1-af} = \phi$ for some $a \in R$, whence $X_{f(1-af)} = \phi$. Hence $f(1-af) \in N$, that is, $(\bar{f}) = (\bar{f}^2)$. Therefore R/N is regular.

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環의 PRIME SPECTRUM에 관하여

金 應 泰

要 約

單位元을 가지는 可換環에 있어서의 Prime Spectrum에 관하여 다음 세가지 事實을 證明하였다.

1. X를 環 R의 prime spectrum, C(X)를 X에서 定義되는 實連續函數의 環, X를 C(X)의 maxima spectrum이라 하면 $\tilde{X}$는 C(X)의 prime spectrum의 部分空間으로서의 한 T—space로 된다. N을 環 R의 nilradical이라 하면, R/N이 regular이면 X와 $\tilde{X}$는 位相同型이다.

2. f: $R \rightarrow R'$ 을 ring homomorphism, P를 R의 한 prime ideal, R_P, R'_P를 各各 $S=R-P$ 및 f(S) 에 관한 分數環(ring of fraction)이라 하고, k(P)를 local ring R_P의 residue' field라 할 때, R'의 prime spectrum의 部分空間인 $f^{*-1}(P)$는 $k(P) \otimes_R R'$의 prime spectrum과 位相同型이다. 단 f^*는 $f^*(Q) = f^{-1}(Q)$로서 정의되는 函數 $f^*: Spec(R') \rightarrow Spec(R)$이다.

3. X를 環 S의 prime spectrum, N을 R의 nilradical이라 할 때, 다음 네가지 事實은 同値이다.

(1) R/N은 regular이다.

(2) X는 Zarski topology에 관하여 Hausdorff 空間이다.

(3) X에서의 Zarski topology와 constructible topology와는 一致한다.

(4) R의 任意의 元素 f에 대하여 f를 包含하지 않는 R의 prime ideal 全體의 集合 X_f는 Zarsk topology에 관하여 開集合인 同時에 閉集合이다.