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ONFUNCTIONS WITH H^P DERIVATIVE

by

Tai-Sung Song

Seoul National University, Seoul, Korea

1. Introduction.

Let D be the open unit disk and H the set of analytic functions f in D with $f(0)=0$, $f'(z)\neq 0$ and $|\arg f'(z)|<\infty$, $f'(0)=1$. Hornich has defined operations on H so that it is a real Banach space [2]. That is if f, g are in H and α is real

$$[f,g](z)=\int_0^z f'(t)g'(t)dt,$$

$$[\alpha\times f](z)=\int_0^z [f'(t)]^\alpha dt,$$

and $||f||=\sup\{|\arg f'(z_1)-\arg f'(z_2)|;\ z_1,\ z_2 \text{ in } D\}$.

In this paper we investigate the relationship of H (as a set of functions) to the Hardy spaces H^p of functions in the Lebesgue space L^p which have zero negative Fourier coefficients.

2. Main theorems.

Given f in H and K a number satisfying

$$\pi K \geqq 2\sup|\arg f'(z)|,\quad z\in D$$

then the function $Q(z)=\exp[K^{-1}\log f'(z)]$ is analytic with positive real part and $Q(0)=1$. The function

$$w(z)=\frac{Q(z)-1}{Q(z)+1}$$

is a H^∞ function satisfying the hypotheses of the Schwarz lemma. Thus for each function f in H there a H^∞ function $w(z)$ ($|w(z)|\leqq 1$, $w(0)=0$) such that

$$f'(z)=\left(\frac{1+w(z)}{1-w(z)}\right)^K$$

Theorem 1. Every function in H has an H^p derivative for some $p<1$.

Proof. Let f be a function in H. Define

$$G(z)=\left(\frac{1+z}{1-z}\right)^K$$

Then G belongs to H^p for all $0<p<1/K$ and $f'(z)=G(w(z))$. Thus f' is subordinate to G, and hence $f'\in H^p$ for all $0<p<1/K$ [1, p. 10]. This completes the proof of theorem.

Since $f'\in H^p$ for some $p<1$, the following result follous from a theorem of Hardy and Littlewood [1, p. 88].

Corollary. For each f in H there is a number $p>0$ such that f is in H^p.

We remark here that for every function in H there is a number $p>0$ such that $|f'(z)|^p$ has a harmonic majorant.

Next, we consider the harmonic function Ref' with $f\in H$. We let $p(r,\theta)$ denote the Poisson kernel;

$$p(r,\theta)=\frac{1-r^2}{1-2r\cos\theta+r^2}$$

Theorem2. For each function f in H the real part of f' can be expressed as the Poisson-Stieljes integral.

Proof. Let $u=Re\,f'$. By theorem 1, we know that

$$\frac{1}{2\pi}\int_0^{2\pi}|u(re^{i\theta})|\,d\theta$$

is finite. Define

$$m_r(t)=\int_0^t u(re^{i\theta})\,d\theta$$

Then the functions $m_r(t)$ are of uniformly bounded variation. By the Helly selection theorem, there is a sequence $\{r_n\}$ tending to 1 for which $m_{r_n}(t)\to m(t)$, a function of bounded variation in $0\leqq t\leqq 2\pi$. Thus

$$u(z)=\lim_{n\to\infty}u(r_nz)=\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}p(r,\theta-t)u(r_ne^{it})\,dt$$

$$=\lim_{n\to\infty}\frac{1}{2\pi}\int_0^{2\pi}p(r,\theta-t)\,dm_{r_n}(t)=\frac{1}{2\pi}\int_0^{2\pi}p(r,\theta-t)\,dm(t).$$

Hornich shows that ball $\{f\in H:\|f\|\pi\}$ contains only univalent functions [2]. Thus we have the following result from a theorem of [1, p. 50].

Theorem 3. Every function f in H with $\|f\|\leqq\pi$ belongs to H^p for all $p<\frac{1}{2}$.

Referencet

1. W.L. Duren, Theory of H^p spaces, Academic-Press, New York, 1970.
2. H.Hornich, Ein Banachraum analytischer Funktionen in Zusammenhang mit den schlichten Funktion, Monat. Math. 73(1969), 36-45.