Integrability In The Usual Topological Space

by

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A. Introduction. The purpose of the present paper is to find the necessary and sufficient conditions for a bounded function $f: R\longrightarrow R$ to be Riemann-integrable or Lebesgue-integrable when we use the concept of filterbase.

B. Notations.

(a) $P=\{x_0, x_1, x_2, \cdots, x_n\}$ denotes a partition of a closed interval $[a, b]$ and $\boldsymbol{P}$ denotes the directed set of all partitions of $[a, b]$.

(b) For a bounded function $f: [a, b]\longrightarrow R$ and a partition $P=\{x_0, x_1, \cdots, x_n\}$ of [a, b], let

$M_k=\sup\{f(x) : x\in[x_{k-1}, x_k]\}$ and $m_k=\inf\{f(x) : x\in[x_{k-1}, x_k]\}$,

$S_p=\sum_{k=1}^{n} M_k(x_k-x_{k-1})$ and $s_p=\sum_{k=1}^{n} m_k(x_k-x_{k-1})$

(c) Let $I=\inf\{S_p : P\in\boldsymbol{P}\}$ and $J=\sup\{s_p : P\in\boldsymbol{P}\}$.

(d) $R_p=\{r : r=\sum_{k=1}^{n} f(\bar{x}_k)\ (x_k-x_{k-1}),\ \bar{x}_k\in[x_{k-1}, x_k],\ k=1, 2, \cdots, n\}$.

(e) Let $A_p=\bigcup\{R_Q : P<Q\}$ and $\boldsymbol{A}=\{A_p : P\in\boldsymbol{P}\}$.

C. A few propositions.

Lemma 1. A bounded function $f: [a, b]\longrightarrow R$ is Riemann-integrable if and only if for any positive real number ε there exists $P\in\boldsymbol{P}$ such that $|S_p-s_p|<\varepsilon$.

Lemma 2. If $P<Q$, then $R_Q\subset R_p$.

Lemma 3. A is a filterbase in the usual topoligical space (R, u).

Proof. $\boldsymbol{A}$ has the following two properties:

(1) For any $P\in\boldsymbol{P}$, $A_p=\bigcup\{R_Q : P<Q\}\neq\phi$.

(2) For any $P, Q\in\boldsymbol{P}$, there exists the cross partition $P\times Q$ such that $A_{P\times Q}\subset A_P\cap A_Q$.

D. The necessary and sufficient conditions for Riemann-integrability

Theorem. A bounded function $f: [a, b]\longrightarrow R$ is Riemann-integrable if and only if $\boldsymbol{A}$ converges.

Proof. Let $f: [a, b]\longrightarrow R$ be Riemann-integrable, then $I=J$. Put $I=J-k$ and for any positive real number $U(k)=(k-\varepsilon,\ k+\varepsilon)$. By Lemma 1 there exists $P\in\boldsymbol{P}$ such that $|S_p-s_p|<\frac{1}{2}\varepsilon$, that is, $s_p\in(k-\frac{1}{2}\varepsilon,\ k+\frac{1}{2}\varepsilon)$ and $S_p(\in k-\frac{1}{2}\varepsilon,\ k+\frac{1}{2}\varepsilon)$ By the definition of R_p, S_p and S_p and S_p, $S_p\in\bar{R}_p$ and $s_p\in\bar{R}_p$, where $\bar{R}_p$ denotes the closure of R_p. Therefore $R_p\in(k-\varepsilon,\ k+\varepsilon)$. Let $A=\bigcup\{R_Q : P<Q\}$, then by Lemma 2 $A_p\subset U(k)$. Hence for any open

set G which contains k, there exists $A \in \boldsymbol{A}$ such that $A \subset G$. This shows that $\boldsymbol{A}$ converges to k. Let A converges to k. Then for any positive real number there exists a partition $P \in \boldsymbol{P}$ such that $A_p \in (k-\frac{1}{4}\varepsilon,\ k+\frac{1}{4}\varepsilon)$ Since $R_p \subset A_p$ and $S_p \in \bar{R}_p$, $s_p \in \bar{R}_p$, we have $S_p \in (k-\frac{1}{2}\varepsilon\ k+\frac{1}{2}\varepsilon)$ and $s_p \in (k-\frac{1}{2}\varepsilon,\ k+\frac{1}{2}\varepsilon)$, that is, $|S_p - s_p| < \varepsilon$. By Lemma 1 f is Riemann-integrable.

E. Other Notaton and proposition.

(a) Let $f:[a,b] \longrightarrow R$ be a bounded and measurable function and α, β be two real number such that $\alpha < f(x) < \beta$ for all $x \in [a,b]$. $\boldsymbol{P}$ denotes the directed set of all partitions of $[\alpha, \beta]$.

(b) $m(E)$ denotes the measure of E.

(c) For a partitions $P = \{y_0, y_1, y_2, \cdots, y_n\}$ of $[\alpha, \beta]$, let

$S_p = \sum_{k=1}^{n} y_k m(E_k)$ and $s_p = \sum_{k=1}^{n} y_{k-1} m(E_k)$, where $E_k = \{x : y_{k-1} \leqq f(x) \leqq y_k\}$.

(d) $I^* = \inf\{S_p : P \in \boldsymbol{P}\}$, $J^* = \sup\{s_p : P \in \boldsymbol{P}\}$.

(e) $L_p = \{r : r = \sum_{k=1}^{n} \bar{y}_k m(E_k),\ \bar{y}_k \in [y_{k-1}, y_k],\ k = 1, 2, \cdots, n\}$.

(f) $B_p = \bigcup \{L_Q : P < Q\}$ and $\boldsymbol{B} = \{B_p : P \in \boldsymbol{P}\}$.

Lemmal. $\boldsymbol{B}$ is a filterbase in (R, U).

Proof. $\boldsymbol{B}$ has the following two properties:

(1) For any $P \in \boldsymbol{P}$, $B_p = \bigcup \{L_Q : P < Q\} \neq \phi$.

(2) For any $P, Q \in \boldsymbol{P}$, there exists the cross partition $P \times Q$ such that $B_{P \times Q} \subset B_P \cap B_Q$.

F. The necessary and sufficient conditions for Lebesgue-integrability

Theorem. A bounded and measurable function $f:[a,b] \longrightarrow R$ is Lebesgue-integrable if and only if $\boldsymbol{B}$ converges.

Proof. Let $f:[a,b] \longrightarrow R$ be Lebesgue-integrable, then $I^* = J^*$. Let $I^* = J^* = k$ and $U(k) = (k-\varepsilon,\ k+\varepsilon)$ for any positive real numberes ε. Then there exists $P \in \boldsymbol{P}$ such that $k-\frac{1}{2}\varepsilon < s \leqq S_p < k+\frac{1}{2}\varepsilon$. By the definition of L_p, s_p and S_p, $s_p \in \bar{L}_p$ and $S \in \bar{I}_p$.

Hence $L_p \subset (k-\varepsilon,\ k+\varepsilon) = U(k)$.

Let $\boldsymbol{B}$ converges to k. Then for any poaitive ε there exists a partition $P \in \boldsymbol{P}$ such that $B_p \subset (k-\frac{1}{2}\varepsilon,\ k+\frac{1}{2}\varepsilon)$. Since $L_p \subset B_p$ and $S_p \in \bar{L}_p$, $s_p \in \bar{L}_p$, we have

$S_p \in (k-\varepsilon,\ k+\varepsilon)$ and $s_p \in (k-\varepsilon,\ k+\varepsilon)$, that is,

$|S_p - k| < \varepsilon$ and $|s_p - k| < \varepsilon$.

Since ε is any positive real number, we have

$\inf\{S_p : P \in \boldsymbol{P}\} = k$ and $\sup\{s_p : P \in \boldsymbol{P}\} = k$.

This shows that f is Lebesgue-integrable.