韓國數學教育學會誌〈數學教育〉
1974. 12 Vol. XIII No. 2

Journal of the Korea Society of Mathematical Education
Dec. 1974. Vol. XIII No. 2

ELASTIC SPACES AND MONOTONICALLY NORMAL SPACES

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1) INTRODUCTION. Recently P. Zenor introduced the class of Monotonically normal spaces. Soon afterward, *R.* Heath and *D.* Lutzer proved that each linearly ordered topological space is monotonically normal. Since Zenor did not know if monotonically normal space were hereditarily monotonically normal and the result of Heath and Lutzer had a very long proof and Carlos *R.* Borges proved the question in [3], each elastic space is monotonic is monotonically normal space, each linearly ordered topological space is monotonically normal and the characterization of monotonically normal space. In [4], Gary Gruenhage proved that monotonically normal space is not elastic with counterexample (Sorgenfrey line is not elastic). In this paper, I have proved that if f is a continuous closed surjection from monotonically normal space X to a topological space Y, then Y is monotonically normal.

2) PRELIMINARIES AND DEFINTION. We will first define monotonically normal space.

DEFINITION 2. 1. For any space X, let $\mathfrak{D}_X=\{(A,B)\,|\,A$ and B are disjoint closed subsets of $X\}$. The T_1-space X is said to be monotonically normal provided that to each $(A,B)\in\mathfrak{D}_X$ one can assign an open subset $G(A,B)$ of X such that

(a) $A\subset G(A,B)\subset G(A,B)^-\subset X\text{-}B$,

(b) $G(A,B)\subset G(A',B')$, whenever $A\subset A'$ and $B'\subset B$.

The function G is called a monotone nomality operator.

PROPOSITION 2. 2. The following are equivalent:

(a) X is monotonically normal

(b) X is T_1, and to each pair (A,B) of subsets of X, with $A^-\cap B=A\cap B^-=\phi$, one can assign an open subset $G(A,B)$ of X such that

(i) $A\subset G(A,B)\subset G(A,B)^-\subset X-B$

(ii) $G(A,B)\subset G(A',B')$ whenever $A\subset A'$ and $B'\subset B$

(c) To each pair (A,U) of subsets of X, with A closed, U open and $A\subset U$, we can assign an open set $U_A\supset A$ such that

(i) $U_A\subset V_B$ whenever $A\subset B$ and $U\subset V$

(ii) $U_A\cap(X-A)_{X-U}=\phi$

(d) For each open $U\subset X$ and $x\in U$ there exists an open neighbourhood U_x of x such that $U_x\cap V_y\neq\phi$ implies $x\in V$ or $y\in U$.

(e) There exists a base $\mathfrak{B}$ for X such that, for each $B \in \mathfrak{B}$ and $x \in B$ there exists an open neighborhood B_x of x such that

$$B_x \cap C_y \neq \phi \text{ implies } x \in C \text{ or } y \in B$$

(f) X satisfies Lemma 2.1 of [5]

(g) X satisfies Lemma 5.1 of [5]

PROOF. [3]

3) MAINTHEOREM

THEOREM 3.1. X is monotonically normal space, and if $f: X \to Y$ is continous and closed surjection, with Y a topological space. Then Y is monotonically normal.

PROOF. For each pair (A, B) of disjoint closed subsets of Y, Since f is closed and surjection, there exist disjoint closed subsests A', B' of X with $f(A') = A$, $f(B') = B$. On the other hand, X is monotonically normal space, so there exists open subset $G(A', B')$ of X such that $A' \subset G(A', B') \subset G(A', B')^- \subset X - B'$ and $G(A', B') \subset G(A'', B'')$ whenever $A' \subset A''$ and $B'' \subset B'$. Here function f is continuous and so there exists open set $H(A, B)$ in Y such that $G(A', B') = f^-(H(A, B))$. For this operator H, we define $H(A, B) = f(G(A', B)) = G(f(A'), f(B'))$. then i) $A \subset H(A, B) \subset H(A, B)^- \subset Y - B$, ii) $H(A, B) \subset H(C, D)$ whenever $A \subset C$ and $D \subset B$ in Y. By the function f and above statement, $f(A') \subset f(G(A', B')) \subset f(G(A', B'))^- \subset f(X - B')$ is obvious and $f(A') = A$, $f(G(A', B')) = H(A, B)$, $G(A', B')^-$ is closed in X, so there exists open subset $G(C, D)$ in X with $G(A', B')^- = X - G(C, D)$, then $f(X - G(C, D)) = Y - f(G(C, D))$

$$\begin{aligned} &= Y - f(U\{V_x \mid x \in C,\ V \subset X - D\}) \\ &= Y - U\{U_{f(x)} \mid f(x) \in f(C),\ U \subset Y - f(D)\} \text{ : closed in } Y \\ &= Y - G(f(C), f(D)) \\ &= H(A, B)^- \text{ for some } H(A, B) = f(G(A', B')). \end{aligned}$$

Hence we have proved $f(G(A', B')^-) = H(A, B)^-$, and $f(X - B')$ is equal to $Y - B$, because generally $f(X - B') \supset Y - B$ and for every $x \in f(X - B')$, there exists y in $X - B'$ with $y = f^{-1}(x)$ by the function f is surjection, and so $y \notin B'$ and $y \in X$, thus $x \notin f(B') = B$ and $x \in f(X) = Y$. Now we have proved i). Now suppose $f(A') = A$, $f(B') = B$, $f(C') = C$ and $f(D') = D$, for some closed sets A, B, C and D in Y, clearly $f(A') \subset f(C')$ and $f(D') \subset f(B')$ by the hypothesis of ii).

$$\begin{aligned} \text{Since } f(G(A', B')) &= f(U\{\mathfrak{U}_x \mid x \in A',\ \mathfrak{U} \subset X - B'\}) \\ &= U\{f(\mathfrak{U}_x) \mid x \in A',\ \mathfrak{U} \subset X - B'\} \\ &= U\{V_{f_x(x)} \mid f(x) \in f(A'),\ V \subset Y - f(B')\} \\ &= G(f(A'), f(B')) = H(A, B) \end{aligned}$$

$$\text{Then} \quad \underset{H(A, B)}{\overset{f(G(A', B')) = G(f(A'), f(B'))}{\|}} \quad \subset G(f((C''), f(D')) = f(G(C', D')) = H(C, D)$$

thus we have proved. the theorem

4) QUESTION. In following diagram all the cases were proofed but I offer the question below

(a) What kind of condition are needed that monotonically normal space to be a elastic space?

(b) What kind of condition are needed that normal space to be nomotonically normal?

Linearly ordered topological space
⇓
Elastic space ⇒ / ⇐/= Monotonically normal space
⇓
Normal space

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요 약

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P.Zenor 에 의해서 Monotonically Normal space 가 정의되었으며 그후 R. Heath 와 D. Lutzer에 의해서 Linearly ordered topological space 가 Monotonically Normal 임을 증명했다. 한편 Zenor 는 Monotonically Normal Space 의 hereditary 에 관한 것을 question 으로 남겼는데 Heath 와 Lutzer가 증명했고 또 그 증명보다 더 간단한 증명을 Calos R. Boyers 가 증명했다(3). 뿐만 아니라 그결과로서 Linearly ordered topological space 와 Elastic space 가 Monotonically Normal space 임을 밝혔다. 또 (4)에서 Gary Gruenhage 가 Monotonically Normal space 가 Elastic space 가 안됨을 counterexample 을 들어서 증명했다. 결론적으로 Monotonically Normal space 와 Elastic space 는 완전히 분리되었다. 또 Elastic space 의 closed continuous image 는 paracompact 이고 Monotonically Normal 임을 증명했다.

이 논문에서 본인이 밝힌 것은 Monotonically Normal space 의 closed continuous image 가 Monotonically Normal 임을 밝혔다.