韓國數學敎育學會誌〈數學敎育〉
1974. 12 Vol. XIII No. 2

Journal of the Korea Society of Mathematical Education
Dec. 1974. Vol. XIII No. 2

# Metrizability of wM-spaces

by

Heung Ki Kim

*Kang Won National University, Chun Cheon, Korea*

This paper concerns with the metrizability of M-spaces with $G\delta$-diagonals. Several results obtained recently on this question are:

1. (Zenor) An M-space is metrizable iff it has a regular $G\delta$-diagonal.
2. (Borges) A regular meta-Lindelöf M-space is metrizable iff it has a $\bar{G}\delta$-diagonal.
3. (Ishii) A wM-space is metrizable iff it has a $\bar{G}\delta(2)$-diagonal.
4. (Ishii) A normal wM-space is metrizable iff it has a $\bar{G}\delta(1)$-diagonal.
5. (Shiraki) A normal wM-space is metrizable iff it is a $\sigma^{\#}$-space.
6. (Martin) A regular space is metrizable iff it is a c-semistratifiable wM-space.

In this paper we will show that for regular spaces all six of the foregoing metrization theorems are special cases of Corollary 5. We adopt the convention that if $\{x_n\}$ is a sequence, $\langle x_n\rangle$ denotes the range of the sequence $\{x_n\}$ and $\langle x;x_n\rangle$ denotes $\{x\}\cup\langle x_n\rangle$.

A cs-semistratification for a topological space $X$ is a mapping $g$ from $N\times X$ to the topology of $X$ which satisfies the following conditions:

CS-1 $x\in g(n,x)$;

CS-2 $g(n+1,x)\subset g(n,x)$;

CS-3 if a sequence $\{x_n\}$ converges to a unique point $x$, then

$$\bigcap_{i=1}^{\infty} g(i,\langle x;x_n\rangle)=\langle x;x_n\rangle.$$

Here, we used the notation that

$$g(n,s)=\cup\{g(n,s):s\in S\}$$

for every subset $S$ of $X$.

A space is said to be cs-semistratifiable if $X$ has a cs-semistratification. A cs-semistratification $g$ is a semistratification(3) if $g$ satisfies the following condition:

(*) $F=\cap\{g(n,F):\ n=1,2,\cdots\}$ for every closed subset $F$ of $X$.

Because, if $U$ is an open subset of a space $X$ which has a cs-semistratification satisfing (*), let $U_n=X-g(n,X-U)$. Then the sequence $\{U_n\}$ satisfies the conditions of (3, Definition1. 1).

Remark. In any topological space, the sequence $\{x,x,x,\cdots\}$ converges to every point in $\mathrm{cl}(x)$, it is clear from CS-3 that

$$\bigcap \{g(n, x):\ n=1, 2, \} = \mathrm{cl}(x)$$

for all $x \in X$.

**THEOREM 1.** A Hausdorff c-semistratifiable space is cs-semistratifiable.

Proof. Let $\{x_n\}$ be a sequence converging to $x_0$ in a Hausdorff c-semistratifiable space $X$. Since $X$ is Hausdorff, the set $\langle x_0;x_n\rangle$ cannot have any other cluster point. This implies that $\langle x_0;x_n\rangle$ is closed and compact. For each $x \notin \langle x_0;x_n\rangle$, there exists a positive integer $k$ such that $x \notin g(k, \langle x_0;x_n\rangle)$, and hences $x \notin \bigcap \{g(i, \langle x_0;x_n\rangle): i=1, 2, \cdots\}$. This insures that $\bigcap \{g(i, \langle x_0, x_n\rangle): i=1, 2, \cdots\} \subset \langle x_0, x_n\rangle$. The reverse inclusion is clear from CS-1.

**COROLLARY 2.** Any space with a $\bar{G}_\delta(1)$-diagonal is cs-semistratifiable.

Proof. If a space has a $\bar{G}_\delta(1)$-diagonal, it is Hausdorff. Now apply Theorem 1 and (10, Theorem 1).

A topological space $X$ is a $\beta$-space provided that there is a mapping $g$ from $N \times X$ to the topology of $X$ such that $x \in g(n, x)$ for all $n$ and all $x$ and if $x \in g(n, x_n)$ for some $x \in X$ and a sequence $\{x_n\}$ in $X$, then $\{x_n\}$ has a cluster point. Hodel proved that a space is semistratifiable iff it is both a $\beta$-space and a $\sigma^{*}$-space(4). Martin proved that a regular space is semistratifiable iff it is a c-semistratifiable $\beta$-space(10). But it remains true when c-semistratifiabilities are replaced by cs-semistratifiabilities.

**THEOREM 3.** A regular space is semistratifiable iff it is a cs-semistratifiable $\beta$-space.

Proof. That a regular semistratifiable space is a cs-semistratifiable $\beta$-space is an easy consequence of Theorem 1 and (10, Theorem 3).

For the converse, let $X$ be a regular cs-semistratifiable $\beta$-space with a cs-semistratification $g$ such that $\mathrm{cl}\, g(n+1, x) \subset g(n, x)$ for all $x \in X$ and all $n$ and such that if $a \in g(n, b_n)$ for $n=1, 2, 3, \cdots$, then the sequence $\{b_n\}$ has a cluster point. Let $x, x_n \in X$ such that $x \in g(n, x_n)$ for $n=1, 2, 3, \cdots$. we will show that $\{x_n\}$ converges to $x$.

The sequence $\{x_n\}$ has at least one cluster point $x$; moreover, every subsequence of $\{x_n\}$ also has at least one cluster point. Suppose $y$ is a cluster point of $\{x_n\}$ distinct from $x$. Choose a-subsequence $\{x_{ni}\}$ of $\{x_n\}$ with $x_{ni} \in g(i, y)$ for $i=1, 2, \cdots$ and $x_{ni} \neq x$ for all $i$. Since $\mathrm{cl}\, g(i+1, y) \subset g(i, y)$, $y$ is the only one cluster point of $\{x_{ni}\}$, it follows that $x_{ni} \to y$ so that there exists an $m$ such that $x \notin g(n, \langle y;x_{ni}\rangle)$ if $n > m$. Take $k > m$. Then $x \notin g(m, x_k) \supset g(k, x_k)$, which is a contra diction. It follows that $x$ is the unique cluster point of $\{x_n\}$. Since every subsequence of $\{x_n\}$ has a cluster point, $\{x_n\}$ converges to $x$. This completes the proof.

**COROLLAYR 4.** A regular space is developable iff it is a cs-semistratifiable w$\Delta$-space.

Proof. Note that any w$\Delta$-space is a $\beta$-space and apply Theorem 3.

**COROLLARY 5.** A regular space is metrizable iff it is cs-semistratifiable wM-space

Proof. Note that every wM-space is a $\beta$-space and apply Theorem 3 and (10, Corollary 5).

## References


1. C.Borges, On metrizability of topoogical spaces, Can. J. Math. 20 (1968), 795-804.
2. D.Burke, On p-spaces and w$\Delta$-spaces, Pac. J. Math. 35(1970), 285-296.
3. G.Creede, Concerning semistratifiable spaces, Pac. J. Math. 32(1970), 47-54.
4. R.E.Hodel, Moore spaces and w$\Delta$-spaces, Pac. J. Math. 38(1971), 641-652.
5. T.Ishii, On wM-spaces. I, Proc. Japan Acad. 46(1970), 6-10.
6. ——, On wM-spaces. II, Proc. Japan Acad. 46(1970), 11-15.
7. H.Martin, Topological spaces in which compacta are uniformly $G_\delta$ (to appear).
8. T.Shiraki, On some metrization theorems (to appear).
9. P.Zenor, On spaces with regular $G_\delta$-diagonals, Pac. J. Math. 40(1972), 759-763.
10. H.Martin, Metrizability of M-spaces, Canad. J.Math. 25(1973), 840-841.
11. J.Kelley, General Topology, Van Nostrand, 1955.