

◀Original▶ Toroidal Equilibrium of Screw Belt Pinch

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Abstract

The equilibrium configuration of the screw belt pinch is obtained as the first-order correction on that of the straight pinch of infinite length.

요 약

스크류 벨트 핀치의 평형 위치가 무한장의 똑바른 핀치의 평형 위치에 대한 일차교정으로써 구해졌다.

1. Introduction

A high β plasma (by β we mean the ratio of the plasma pressure to the pressure of outside magnetic field just after the implosion) is not only required for controlled fusion but has a special advantage for it can be heated easily to thermonuclear temperatures in a short time. From this point of view, several attempts¹⁻⁴⁾ to confine high β toroidal plasma with circular cross section have been made. But for such high β toroidal plasmas of circular cross section, the toroidal current which is necessary for the equilibrium has to exceed a critical value⁵⁾ which determines the stability against kinks, wavelengths of which are less than the circumference of the torus. This difficulty might be overcome by changing the plasma cross section from a circular one to a strongly elongated one. The idea for this is

that the equilibrium can exist without the toroidal current (poloidal field) when the plasma configuration has no z -dependency ($\frac{\partial}{\partial z} = 0$) where the z -direction is the direction of the axis of torus⁶⁾. However, for this non-toroidal current configuration, the center of the plasma cross section might be located too close to the wall, and $\frac{\partial}{\partial z} \neq 0$ at the ends. For these reasons, the so-called belt pinch⁷⁾ has a weak toroidal current in order to give a restoring force against the drift of plasma toward the wall (the "screw" magnetic field).

In the present paper we discuss the existence of toroidal equilibrium and calculate the distance of the center of the plasma column from the center of the tube cross section for any β of the screw belt pinch.

We use the vacuum model⁸⁾ in which the plasma column with a sharp boundary is surrounded by the vacuum. The toroidal effects are taken into account as the first-order cor-

rection on the straight pinch quantities. Here, we assume that the conducting shells have a thickness greater than the skin depth associated with frequency of the main capacitor bank current.

2. Theory

We introduce a toroidal coordinate system as shown in Fig. 1. The coordinates r, ϕ, z are related to the cylindrical coordinates by $r = R - R_0$, where R_0 is the distance of the center of the cross section of the tube from the axis of symmetry.

Let the initial plasma containing a toroidal (ϕ -direction) bias field be imploded by the B_ϕ and B_z field. During the implosion phase, some magnetic field is allowed to diffuse into the plasma due to the anomalous resistivity of the boundary region produced by the current-driven micro-instabilities⁹⁾ But still the force equation is valid in the form

$$\vec{f} = -\Delta p + \frac{1}{c} (\vec{j} \times \vec{B}) \quad (1)$$

during and after implosion¹⁰⁾.

The equations for local force equilibrium after the implosion are

$$f_r = 0, \quad (2)$$

$$f_z = 0, \quad (3)$$

and the Maxwell equations give

$$\vec{j} = \frac{c}{4\pi} \Delta \times \vec{B} \quad (4)$$

$$\text{and } \Delta \cdot \vec{B} = 0 \quad (5)$$

for the rotationally symmetric configuration.

For the belt-pinch configuration as shown in Fig. 1, we then have

$$\frac{\partial}{\partial R} \left(P + \frac{1}{8\pi} B_z^2 + \frac{1}{8\pi} B_\phi^2 \right) = -\frac{B_\phi^2}{4\pi R} \quad (6)$$

$$\frac{\partial}{\partial z} \left(P + \frac{1}{8\pi} B_z^2 \right) = 0 \quad (7)$$

$$\frac{\partial B_z}{\partial z} = 0 \quad (8)$$

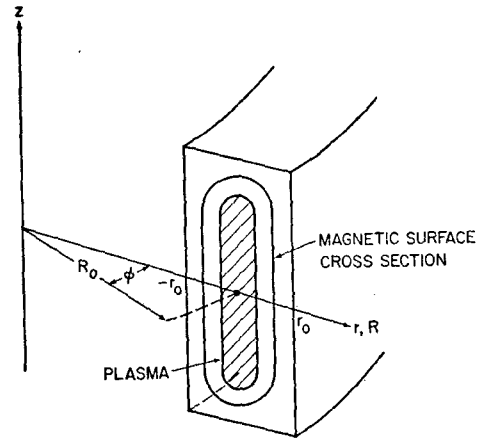


Fig. 1. Belt Pinch Configuration

neglecting the end regions.

From Eqs. (6)-(8), we can easily prove that the equilibrium can be established only when P, B_ϕ, B_z have no z -dependency, which is similar to the case of the actual belt-pinch configuration except for the end regions.

In the following, we will find the location of the center of the plasma in the equilibrium for the belt-pinch configuration as a first-order approximation on the straight pinch quantities. From Eqs. (6)-(8) we have the single equation

$$\frac{d}{dr} \left(P + \frac{1}{8\pi} B_z^2 \right) = -\frac{1}{4\pi \left(1 + \frac{r}{R_0} \right)} \quad (9)$$

$$\frac{d}{dr} \left[\left(1 + \frac{r}{R_0} \right) B_\phi \right] B_\phi$$

where we used the r, ϕ, z coordinate system.

When R approaches infinity, this equation becomes that for the straight pinch of infinite length. Since the order of r/R_0 is less than $1/30$ for the practical belt-pinch⁷⁾, we consider this as the zero-order equation and write all P_0, B_ϕ, B_z in the form

$$P(r) = P^0(r) + p^1(r) \quad (10)$$

$$B_\phi(r) = B_\phi^0(r) + B_\phi^1(r) \quad (11)$$

$$B_z(r) = B_z^0(r) + B_z^1(r) \quad (12)$$

where $P^0(r), B_\phi^0(r), B_z^0(r)$ is a solution of

the zero-order equation, *i.e.*,

$$-\frac{d}{dr} \left(P^0 + \frac{1}{8\pi} B_\phi^{02} + \frac{1}{8\pi} B_z^{02} \right) = 0, \quad (13)$$

and $P^1(r)$, $B_\phi^1(r)$, $B_z^1(r)$ is a first-order correction, which vanishes for $r/R_0 \rightarrow 0$. Substituting Eqs. (10)–(12) into the first-order equation of Eq. (9), we obtain

$$\begin{aligned} \frac{dp^1}{dr} = & -\frac{1}{4\pi} \left(B_z^0 \frac{dB_z^1}{dr} + \frac{dB_z^0}{dr} B_z^1 + B_\phi^0 \right. \\ & \left. \frac{dB_\phi^1}{dr} + \frac{dB_\phi^0}{dr} B_\phi^1 + \frac{B_\phi^{02}}{R_0} \right). \end{aligned} \quad (14)$$

Let the magnetic surface at r of equilibrium in the belt-pinch be moved to r' of equilibrium in the straight pinch by the displacement $\xi(r)$, *i.e.*,

$$r' = r + \xi(r) \quad (15)$$

as we straighten the toroidal shape to the straight one, and this straightening process be adiabatic. Then we can write $B_z(r)$ of the belt-pinch in terms of the $B_z^0(r)$ and $\xi(r)$:

$$\begin{aligned} B_z(r) &= B_z^0(r') \frac{dr'}{dr} = B_z^0(r') \left(1 + \frac{d\xi}{dr} \right) \\ &= B_z^0(r) + \xi \frac{dB_z^0}{dr} + B_z^0(r) \frac{d\xi}{dr} \\ &\quad + \xi \frac{d\xi}{dr} \frac{dB_z^0}{dr} \end{aligned}$$

from which with (12)

$$B_z^1(r) = \xi \frac{dB_z^0}{dr} + B_z^0(r) \frac{d\xi}{dr} \quad (16)$$

where we neglected $\xi \frac{d\xi}{dr} \frac{dB_z^0}{dr}$ as a second-order term. Similarly, we obtain

$$B_\phi^1(r) = \xi \frac{dB_\phi^0}{dr} + B_\phi^0(r) \frac{d\xi}{dr}. \quad (17)$$

Since the straightening process is adiabatic, we have

$$P(r) \rho(r)^{-\gamma} = P^0(r') \rho^0(r')^{-\gamma} \quad (18)$$

where ρ is the density of the plasma, and γ is the ratio of specific heats. Assuming the conductivity of the plasma is kept infinite during the straightening process, we have

$$\rho(r) = \rho^0(r') \frac{dr'}{dr} \quad (19)$$

From Eqs. (18) and (19), and using the same

procedure as for B_z^1 , we can derive

$$P^1(r) = \xi \frac{dP^0}{dr} + r P^0 \frac{d\xi}{dr}. \quad (20)$$

Substituting Eqs. (16), (17) and (20) into Eq. (14), and using Eq. (13), we get the first-order equation for $\xi(r)$ as

$$\begin{aligned} \left[r P^0 + \frac{1}{4\pi} (B_\phi^{02} + B_z^{02}) \right] \frac{d^2 \xi}{dr^2} + (r-2) \\ \frac{dP^0}{dr} \frac{d\xi}{dr} + \frac{1}{4\pi} \left[\left(\frac{dB_\phi^0}{dr} \right)^2 + \left(\frac{dB_z^0}{dr} \right)^2 \right] \xi \\ + \frac{B_\phi^{02}}{4\pi R_0} = 0. \end{aligned} \quad (21)$$

The equilibrium plasma in the straight pinch made by the above adiabatic straightening process will be identical to the equilibrium plasma which is made by imploding the initial plasma in the straight pinch. An idealized equilibrium distribution of P^0, B_ϕ^0, B_z^0 after the implosion of an initial plasma in the straight pinch can be written as

$$\begin{aligned} B_\phi^0(r) &= B_i, \quad B_z^0 = 0, \quad P^0(r) = P_o \quad \text{for } |r| \leq r_1, \\ B_\phi^0(r) &= B_r, \quad |B_z^0(r)| = B_p, \\ P^0(r) &= 0 \quad \text{for } r_1 < |r| \leq r_o \end{aligned} \quad (22)$$

with the pressure balance relationship at the boundary, *i.e.*,

$$P_o^2 + \frac{1}{8\pi} B_i^2 = \frac{1}{8\pi} (B_r^2 + B_p^2) \quad (23)$$

where r_o, r_i are the tube and the plasma half lengths in the r -direction, respectively⁹⁾.

Substitution of Eq. (22) yields the differential expression for the displacement $\xi(r)$, which relates the position of any magnetic surface in the equilibrium plasma made by the implosion of an initial plasma in the toroidal pinch to the position of the same surface if the pinch were straight (the coil turns per unit length, current, and the initial plasma are same for both pinches), as

$$\begin{aligned} \frac{d^2 \xi}{dr^2} + \frac{B_i^2}{(4\pi r P_o + B_i^2) R_o} = 0 \quad \text{for } |r| \leq r_1, \\ \frac{d^2 \xi}{dr^2} + \frac{B_r^2}{(B_r^2 + B_p^2) R_o} = 0 \quad \text{for } r_1 < |r| \leq r_o \end{aligned} \quad (24)$$

The solution of these equations with the phy-

sical condition that $\xi(r)$ and $\frac{d\xi}{dr}$ should be continuous functions of r , and the boundary condition that ξ vanishes at the wall is given as

$$\xi(r) = \begin{cases} -\frac{1}{R_0} [B(r^2 - r_0^2) + 2(A-B)r_1(r-r_0)] & \text{for } r_1 < r \leq r_0 \\ -\frac{1}{R_0} [Ar^2 - Br_0^2 + (A-B)(r_1 - 2r_0)r_1] & \text{for } -r_1 < r \leq r_1 \\ -\frac{1}{R_0} [B(r^2 - r_0^2) - 2(A-B)r_1(r+r_0)] & \text{for } -r_0 < r \leq -r_1 \end{cases} \quad (25)$$

where

$$A = \frac{(1-\beta)}{2+(r-2\beta)} \quad (26)$$

$$B = \frac{\mu}{2(1+\mu)}, \quad (27)$$

$$\beta = \frac{8\pi P_0}{B_r^2 + B_p^2}, \text{ and}$$

$$\mu = \frac{B_r}{B_p} = \text{slope of magnetic line.}$$

It should be noted here that the symmetry of ξ with respect to r comes from the first-order approximation for ξ , and this can be accepted since $|\xi| \ll r_0$.

If we take the maximum of ξ for the first-order approximate value of the displacement of the plasma column with respect to the tube center, Δ (corresponding ξ at $r=0$), we get

$$\Delta = \frac{r_0^2}{R_0} \left\{ \frac{\mu}{2(1+\mu)} + \left[\frac{\mu}{2(1+\mu)} - \frac{1-\beta}{2+(r-2)\beta} \right] \left(\frac{r_1^2}{r_0^2} - \frac{2r_1}{r_0} \right) \right\} \quad (28)$$

From Eq. (28), we note that the displacement Δ of the plasma column in the tube increases

with increasing β for high μ , but remains pleasantly very small for any value of β , *i. e.*,

$$\Delta < \frac{\mu}{2(1+\mu)} \left(1 - \frac{r_1}{r_0} \right)^2 \frac{r_0^2}{R_0} \quad (29)$$

for the actual high μ belt pinch⁷⁾.

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