Some Notes on the Fourier Series of an Almost Periodic Weakly Stationary Process

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0. In my former paper [3] I defined an almost periodicity of weakly stationary random processes (a.p.w.s.p.) and presented some basic results of it. In this paper I shall present some notes on the Fourier series of an a.p.w.s.p., resulting from [3]. All the conditions at the introduction of [3] are assumed to hold without repeating them here. The essential facts are as follows:

The weakly stationary process $X(t,\omega)$, $t\in(-\infty,\infty)$, $\omega\in\Omega$, defined on a probability space (Ω, a, P), has a spectral representation

$$X(t,\omega)=\int_{-\infty}^{\infty} e^{it\lambda}\xi(d\lambda,\omega),$$

where $\xi(\lambda)$ is a random measure. Then, the continuous covariance $\rho(\mu)=E(X(t+u)\ X(t))$ has the form

$$\rho(u)=\int_{-\infty}^{\infty} e^{iu\lambda}\,F(d\lambda),$$

where the spectral distribution function $F(\lambda)$ is related to $\xi(\lambda)$ as $E|\xi(\lambda+0)-\xi(\lambda-0)|^2=F(\lambda+0)-F(\lambda-0)$ $\lambda\in(-\infty,\ \infty)$, assuming that $\rho(u)$ is a uniformly almost periodic function.

1. For Fourier exponents Λ_n, $n=1,2,\cdots$, Fourier coefficients of $X(t,\ \omega)$ become

$$a(\Lambda_n)=\xi(\Lambda_n+0)-\xi(\Lambda_n-0),\ n=1,2,\cdots,$$

as mentioned in [3], the proof of which we shall give here. To do so we shall utilize the following theorem:

$\eta(t,\tau)$, $\varepsilon(t)$ are Borel functions such that

$$\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty}|\eta(t,\tau)|^2 dtF(d\tau)<\infty,$$

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$$\frac{1}{2T}\int_{-T}^{T}|\varepsilon(t)|^2dt<\infty,$$

and

$$f_1(\tau)=\lim_{T\to\infty}g_1(\tau)$$

exist in the sense of convergence in $L_2\{F\}$,
where

$$g_1(\tau)=\frac{1}{2T}\int_{-T}^{T}\varepsilon(t)\eta(t,\tau)dt.$$

Then

$$\text{(i)}\quad \frac{1}{2T}\int_{-T}^{T}\varepsilon(t)\int_{-\infty}^{\infty}\eta(t,\tau)\xi(d\tau)dt=\int_{-\infty}^{\infty}g_1(\tau)\xi(d\tau)$$

$$\text{(ii)}\quad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\varepsilon(t)\int_{-\infty}^{\infty}\eta(t,\tau)\xi(d\tau)dt=\int_{-\infty}^{\infty}f_1(\tau)\xi(d\tau).$$

The proof of this theorem is substantially the same as the proof given in Lemmas 4 and 5 of [2].

Now we shall prove the following:

$$a(\lambda)=\begin{cases}\xi(\Lambda_n+0)-\xi(\Lambda_n-0) & \text{if } \lambda=\Lambda_n,\ n=1,2,\cdots\\ 0 & \text{if } \lambda\neq\Lambda_n,\ n=1,2,\cdots\end{cases}$$

Proof

$$a(\lambda)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}X(t)e^{-i\lambda t}\,dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}e^{-i\lambda t}\int_{-\infty}^{\infty}e^{it\tau}\,\xi(d\tau)dt.$$

To utilize the above theorem we put

$$e^{-i\lambda t}=\varepsilon(t)$$

$$e^{it\tau}=\eta(t,\tau)$$

Then all the conditions of the theorem are satisfied and the conclusions of it hold. The left hand side of (ii) is $a(\lambda)$. The right hand side of (ii) is

$$\int_{-\infty}^{\infty}f_1(\tau)\xi(d\tau)$$

where

$$f_1(\tau)=\lim_{T\to\infty} g_1(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{-i\lambda t}\ e^{i\tau t}\ dt$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{i(\tau-\lambda)t}\ dt$$

$$=\begin{cases} 1 & \text{when } \lambda=\tau \\ 0 & \text{when } \lambda\neq\tau \end{cases}$$

i.e.

$$\int_{-\infty}^{\infty} f_1(\tau)\xi(d\tau)=\xi(\lambda+0)-\xi(\lambda-0)$$

Therefore if $\lambda=\Lambda_n$, $n=1,2,\cdots$, then

$$a(\Lambda_n)=\xi(\Lambda_n+0)-\xi(\Lambda_n-0);$$

if $\lambda\neq\Lambda_n$, $m=1,2,\cdots$, then

$$a(\lambda)=0$$ **q.e.d.**

2. If Λ_n, $n=1,2,\cdots\cdots$, are Fourier coefficients of a uniformly almost periodic function $\rho(u)$ the Parseval equation (*i.e.* Bohr's Fundamental Theorem)

$$\sum_{n\geqslant 1}|A_n|^2=M\{|\rho(u)|^2\}$$

holds. The proof of it in [1] is complicated, but if $\rho(u)$ is a covariance of an a. p. w. s. p. then the proof becomes simple as follows:

Proof

$$M\{|\rho(u)|^2\}=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|\int_{-\infty}^{\infty} e^{i\lambda u}\ F(d\lambda)\right|^2 du$$

$$=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\lambda-\mu)u}\ F(d\lambda)\ F(d\mu)du$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} e^{i(\lambda-\mu)u}\ du\right) F(d\lambda)\ F(d\mu)$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(d\lambda)\ F(d\lambda)$$

$$= \sum_{n=1}^{\infty} | F(\Lambda_{n+0}) - F(\Lambda_{n-0}) |^2$$

$$= \sum_{n=1}^{\infty} | \Lambda_n |^2$$

The last two equalities follow from [3]. **q.e.d.**

REFRENCES

[1] Besicovitch, A. S., *Almost Periodic Functions*, Cambridge: Cambridge Univ. Press, 1932.

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