同心形 구멍을 가진 複合실린더의 過渡的 溫度分布, 熱應力 및 熱變形度의 解析

全 義 進* · 金 曉 哲**

Transient Temperature Distribution, Thermal Stresses and Strains in a Composite Cylinder with a Concentric Hole

by

Euijin Jun*, Hyochul Kim**

要 約

열박음(shrink fitting)으로 因한 同心形 구멍을 가진 複合실린더의 過渡的 溫度分布, 熱應力 및 熱變形度를 理論解析하였다. 溫度分布解析에서 外部 실린더는 均一溫度로 加熱되어, 室溫의 內部 실린더와 接觸面에서 일어나는 熱傳導에 依하여 冷却되고, 外部 表面은 大氣中에 露出된 狀態로 取扱하였다. 熱應力은 平面變形度條件을 滿足하는 것으로 생각하였으며, 物性은 溫度에 無關한 常數로 取扱하였다.

溫度分布는 熱傳導問題만을 考慮함으로서도 有效한 解를 얻을 수 있으며 熱應力은 接觸面에서부터 形成되며, 半經方向應力은 時間이 經過함에 따라 壓縮應力이 增加하여 接觸面에서 最大値를 갖고, 圓周方向應力은 接觸面에서 初期부터 거의 最終狀態와 같은 크기를 갖음을 알 수 있었다. 均一溫度分布가 이루어지면 熱應力의 形成은 完了되게 되며, 이때의 熱應力의 크기와 分布傾向은 平面應力條件을 使用하였다는 事實을 考慮하면 Lamé의 理論解와 一致함을 알 수 있었다.

Nomenclature

E	modulus of elasticity
h	heat transfer coefficient
$J_n(x)$	Bessel function of the first kind, of order n
k	thermal conductivity
q	heat transfer rate
r	radial distance
r_1, r_2, r_3	radii of inner and outer cylinders
t	time
$T_0, T_1, T,$	temperature
$u, v, w,$	radial, tangential, and axial displacement
$Y_n(x)$	Bessel function of the second kind, of order n
α	thermal linear expansion coefficient
β	thermal diffusivity
$\epsilon_r, \epsilon_\theta, \epsilon_z$	radial, tangential and axial strain
λ_j	characteristic values
ν	Poisson's ratio
$\sigma_r, \sigma_\theta, \sigma_z$	radial, tangential and axial stresses
ϕ	stress function

Received: March 28, 1974

* Graduate School, Seoul National University

** Member, Seoul National University

1. Introduction

The unequal temperature distribution in a continuous body will disturb the free expansion of the elements and the thermal stresses are happened. The problem of determining the thermal stress in an elastic body due to a given temperature distribution can be found at many practical applications in machine element design, such as in the design of turbines, jet engines and nuclear reactors.

To determine the temperature distribution is prerequisite problem to be solved for the thermal stresses. It has been studied that the temperature terms in the thermal stress equation could be put into the body force and surface force terms of the ordinary equations of elasticity theory, whence the general thermal stress problem can be included in the conventional theory of elasticity. [8]

Material properties such as modulus of elasticity, thermal linear expansion coefficient are actually dependent upon temperature. Material properties at elevated temperatures are yet remaining problems to be determined except a few materials. It has been studied that reliable solution can be obtained with constant material properties under 500 °F temperature difference for aluminum alloy 2024 T3[6], for low carbon steel [4], and stainless steel. [5]

Generally, the analytic solution for thermal stress can not be obtained if material properties are arbitary functions of temperature except for special cases. [2]

Lamé's thick cylinder theory is often applied to analyze the residual stress distribution in a reinforced cylinder, considering the working pressure and radial interference. But transient state of thermal stress distribution can not be covered by Lamé's theory.

In the work reported here, analytic solution of transient temperature distribution, thermal stresses and strains in a composite cylinder with a concentric hole due to shrink fitting is analyzed with constant material properties in plane strain problem.

2. Analysis of Problem

As a analytic model for temperature distribution, thermal stresses and strains for the engineering purpose, the following assumptions and simplifications are taken.

1) The material properties of both cylinders are identical each other and isotropic.
2) Material properties are independent of temperature.
3) The inner surface of inner cylinder can be regarded as insulated.
4) Temperature drop and resistance at the interfering surface is disregarded.
5) Surface roughness at the interfering surface is disregarded.
6) Propagation condition of the thermoelastic stress and strain are not regarded.

2.1 Temperature Distribution

Considering a composite cylinder shown in Fig.1, the governing equation for temperature T from Fourier's equation can be written as a dimensionless one for the convenience in the analysis.

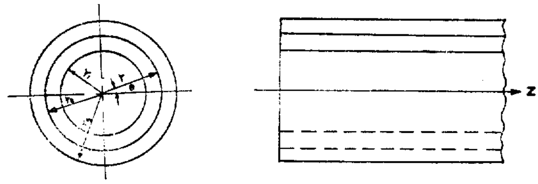

Fig. 1. Geometry and Coordinate System

Let
$$\theta=\frac{T-T_0}{T_1-T_0} \qquad R=\frac{r}{r_3-r_1}$$
$$\tau=\frac{\beta}{(r_3-r_1)^2}t \qquad R_1=\frac{r_1}{r_3-r_1}$$
$$\mu=\frac{h(r_3-r_1)}{k} \qquad R_2=\frac{r_2}{r_3-r_1} \tag{1}$$
$$R_3=\frac{r_3}{r_3-r_1}$$

Then the equation governing θ becomes,

$$\frac{\partial^2\theta}{\partial R^2}+\frac{1}{R}\frac{\partial\theta}{\partial R}=\frac{\partial\theta}{\partial\tau} \tag{2}$$

The outer cylinder is preheated uniformly so that it just slips over the inner cylinder. Let the temperature of outer cylinder T_1, when the inner radius of outer cylinder coincides with the outer radius of inner cylinder as the outer cylinder cools down, the boundary and initial conditions are,

1) at $\tau=0$ $\quad R_1\leq R\leq R_2 \quad \theta=0$ (3)
$\qquad R_2\leq R\leq R_3 \quad \theta=1$

2) $\quad R=R_1 \quad \frac{\partial\theta}{\partial R}=0$ (4)

3) $\quad R=R_3 \quad \frac{\partial\theta}{\partial R}=-\mu\theta$ (5)

The initial temperature distribution is shown in Fig. 2.

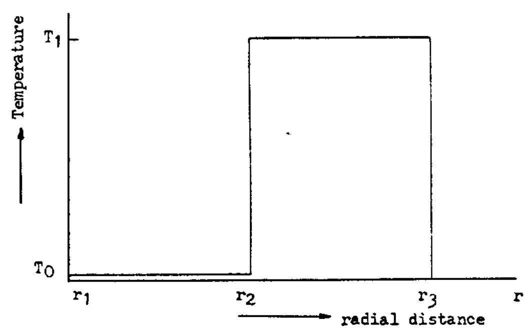

Fig. 2. Initial Temperature Distribution

Suppose that the solution of equation (2) has a form as following,

$$\theta(R,\tau)=F(R)\times\exp(-\lambda^2\tau) \tag{6}$$

where λ is a positive constant.

The solution of the Bessel function of order 0 becomes,

$$F(R)=A(\lambda)J_0(\lambda R)+B(\lambda)Y_0(\lambda R) \quad (7)$$

With the boundary conditions (4) and (5), the following characteristic equation can be obtained.

$$\lambda[J_1(\lambda R_1)Y_1(\lambda R_3)-J_1(\lambda R_3)Y_1(\lambda R_1)]$$
$$-\mu[J_1(\lambda R_1)Y_0(\lambda R_3)-J_0(\lambda R_3)Y_1(\lambda R_1)]=0 \quad (8)$$

Let λ_j be one of the solutions of equation (8), then the solution of equation (7) corresponding to λ_j becomes,

$$F(R)_j=A(\lambda_j)\left[\frac{J_0(\lambda_j R)}{J_1(\lambda_j R_1)}-\frac{Y_0(\lambda_j R)}{Y_1(\lambda_j R_1)}\right] \quad (9)$$

The general solution of equation (2), corresponding to boundary condition(4) and (5) are the linear combinations of $F(R)_j\times\exp(-\lambda_j\tau)$.

Hence

$$\theta=\sum_{j=1}^{\infty}A(\lambda_j)\left[\frac{J_0(\lambda_j R)}{J_1(\lambda_j R_1)}-\frac{Y_0(\lambda_j R)}{Y_1(\lambda_j R_1)}\right]\exp(-\lambda_j^2\tau) \quad (10)$$

$A(\lambda_j)$ can be determined with the initial condition (3), using the orthogonalities of F_j's.

$$A(\lambda_j)=\frac{2}{\lambda_j}\left\{R_3\left[\frac{J_1(\lambda_j R_3)}{J_1(\lambda_j R_1)}-\frac{Y_1(\lambda_j R_3)}{Y_1(\lambda_j R_1)}\right.\right.$$
$$\left.\left.-R_2\left[\frac{J_1(\lambda_j R_2)}{J_1(\lambda_j R_1)}-\frac{Y_1(\lambda_j R_2)}{Y_1(\lambda_j R_1)}\right]\right]\right\}\Big/$$
$$\left(R_3^2\left[\left[\frac{J_1(\lambda_j R_3)}{J_1(\lambda_j R_1)}-\frac{Y_1(\lambda_j R_3)}{Y_1(\lambda_j R_1)}\right]^2\right.\right.$$
$$\left.+\left[\frac{J_0(\lambda_j R_3)}{J_1(\lambda_j R_1)}-\frac{Y_0(\lambda_j R_3)}{Y_1(\lambda_j R_1)}\right]^2\right]$$
$$\left.-R_1^2\left[\frac{J_0(\lambda_j R_1)}{J_1(\lambda_j R_1)}-\frac{Y_0(\lambda_j R_1)}{Y_1(\lambda_j R_1)}\right]^2\right) \quad (11)$$

The final temperature distribution is

$$T(r,t)=T_0+(T_1-T_0)\sum_{j=1}^{\infty}A(\lambda_j)\left(\frac{J_0\left(\lambda_j\frac{r}{r_3-r_1}\right)}{J_1\left(\lambda_j\frac{r_1}{r_3-r_1}\right)}-\frac{Y_0\left(\lambda_j\frac{r}{r_3-r_1}\right)}{Y_1\left(\lambda_j\frac{r_1}{r_3-r_1}\right)}\right)\exp\left[-\lambda_j^2\frac{\delta t}{(r_3-r_1)^2}\right] \quad (12)$$

2.2 Thermal Stresses and Strains

The stresses of a cylinder with radial temperature variation across the cylinder wall constitute a complicated two-dimensional problem even within the range of elastic theory since the stress distribution depends both on the axial and radial distance.

To simplify the problem to a one dimensional one, two limiting cases may be considered. The one is to assume that the cylinder is infinitely long so that the strain is independent of axial distance and the plane strain theory becomes valid. Especially if the cylinder has free ends, the so-called generalized plane strain theory can be applied. This affords a good approximation of the stress distribution in the central section of a long cylinder. The other is to assume that the cylinder is infinitely short or becomes virtually a thin disk, so that the plane stress theory becomes valid. The stresses are assumed to be independent of axial distance. Above two cases will be considered respectively.

2.2.1 Plane Strain Problem

The ordinary stress-strain relation in elasticity must be modified for the strain is partly due to thermal expansion, partly due to stress. Shear stresses and strains are zero on account of rotational symmetry in an isotropic material. And the temperature distribution is a function of the radial distance r only. Firstly plane strain problem i,e. $\epsilon_z=0$ will be analyzed and the results will be modified for the solution of the case with free end, i,e. $\epsilon_z=$constant.

a. Inner cylinder

The stress-strain relations in cylindrical coordinate are,

$$\epsilon_r=\frac{1}{E}[\sigma_r-\nu(\sigma_\theta+\sigma_z)]+\alpha(T-T_0)$$
$$\epsilon_\theta=\frac{1}{E}[\sigma_\theta-\nu(\sigma_r+\sigma_z)]+\alpha(T-T_0) \quad (13)$$
$$\epsilon_z=\frac{1}{E}[\sigma_z-\nu(\sigma_r+\sigma_\theta)]+\alpha(T-T_0)$$

Since $\epsilon_z=0$, the third of equation (13) gives

$$\sigma_z=\nu(\sigma_r+\sigma_\theta)-\alpha E(T-T_0) \quad (14)$$

The stress function ϕ can be obtained in making use of the equation of equilibrium and compatibility equation in rotational symmetry with no body force.

$$\phi=-\frac{\alpha E}{(1-\nu)r}\int_{r_1}^{r}Trdr+\frac{A_1}{2}r+\frac{B_1}{r} \quad (15)$$

The stress components are,

$$r_1\leq r\leq r_2$$
$$\sigma_r=\frac{-\alpha E}{(1-\nu)r^2}\int_{r_1}^{r}Trdr+\frac{A_1}{2}+\frac{B_1}{r^2}$$
$$\sigma_\theta=\frac{\alpha E}{(1-\nu)}\left(-T+\frac{1}{r^2}\int_{r_1}^{r}Trdr\right)+\frac{A_1}{2}-\frac{B_1}{r^2} \quad (16)$$
$$u=\frac{(1+\nu)\alpha}{(1-\nu)r}\int_{r_1}^{r}Trdr+\frac{(1+\nu)(1-2\nu)rA_1}{2E}$$
$$-\frac{(1+\nu)B_1}{Er}-\alpha(1+\nu)rT_0$$

b. Outer cylinder

For the outer cylinder has been prestrained on account of initial temperature, the stress-strain relation will be modified as following.

$$\epsilon_r=\frac{1}{E}[\sigma_r-\nu(\sigma_\theta+\sigma_z)]+\alpha(T-T_0)-\alpha(T_1-T_0)$$
$$\epsilon_\theta=\frac{1}{E}[\sigma_\theta-\nu(\sigma_r+\sigma_z)]+\alpha(T-T_0)-\alpha(T_1-T_0)$$
$$\epsilon_z=\frac{1}{E}[\sigma_z-\nu(\sigma_r+\sigma_\theta)]+\alpha(T-T_0)-\alpha(T_1-T_0) \tag{17}$$

By similar procedure with the case of inner cylinder, the stresses and radial displacement are,

$r_2\leqq r\leqq r_3$

$$\sigma_r=-\frac{\alpha E}{(1-\nu)r^2}\int_{r_2}^{r}Trdr+\frac{A_2}{2}+\frac{B_2}{r_2}$$
$$\sigma_\theta=\frac{\alpha E}{1-\nu}\left(-T+\frac{1}{r^2}\int_{r_2}^{r}Trdr\right)+\frac{A_2}{2}-\frac{B_2}{r^2} \tag{18}$$
$$u=\frac{(1+\nu)\alpha}{(1-\nu)r}\int_{r_2}^{r}Trdr+\frac{(1+\nu)(1-2\nu)rA_2}{2E}-\frac{(1+\nu)B_2}{Er}-\alpha(1+\nu)rT_1$$

Constants A_1, A_2, B_1, and B_2 can be completely determined by following boundary conditions.

Boundary conditions

1) External forces are not applied at the inner surface of inner cylinder and outer surface of outer cylinder,
2) At the interfering surface $r=r_2$, the radial displacement and relative forces of two cylinders are same,

From the above boundary conditions, and equation (16), (18), the constants of integration A_1, A_2, B_1, and B_2 are

$$A_1=\frac{\alpha E}{(1-\nu)(r_3^2-r_1^2)}\left[2\int_{r_1}^{r_3}Trdr-(r_3^2-r_2^2)(T_1-T_0)\right]$$
$$B_1=\frac{\alpha E}{(1-\nu)(r_3^2-r_1^2)}\left[-r_1^2\int_{r_1}^{r_3}Trdr+\frac{r_1^2(r_3^2-r_2^2)}{2}(T_1-T_0)\right]$$
$$A_2=\frac{\alpha E}{(1-\nu)(r_3^2-r_1^2)}\left[2\int_{r_1}^{r_3}Trdr+(r_2^2-r_1^2)(T_1-T_0)\right]$$
$$B_2=\frac{\alpha E}{(1-\nu)(r_3^2-r_1^2)}\left[-r_3^2\int_{r}^{r_2}Trdr-r_1^2\times\int_{r_2}^{r_3}Trdr-2r_3^2(r_2^2-r_1^2)(T_1-T_0)\right] \tag{19}$$

Hence the stress components are,

Inner cylinder

$$\sigma_r=\frac{\alpha E}{1-\nu}\left[-\frac{1}{r^2}\int_{r_1}^{r}Trdr+\frac{r^2-r_1^2}{(r_3^2-r_1^2)r^2}\times\int_{r_1}^{r_3}Trdr-\frac{(r_3^2-r_2^2)(r^2-r_1^2)}{2(r_3^2-r_1^2)r^2}(T_1-T_0)\right]$$
$$\sigma_\theta=\frac{\alpha E}{1-\nu}\left[-T+\frac{1}{r^2}\int_{r_1}^{r}Trdr+\frac{r^2+r_1^2}{(r_3^2-r_1^2)r^2}\times\int_{r_1}^{r_3}Trdr-\frac{(r_3^2-r_2^2)(r^2+r_1^2)}{2(r_3^2-r_1^2)r^2}(T_1-T_0)\right] \tag{20}$$

Outer cylinder

$$\sigma_r=\frac{\alpha E}{1-\nu}\Big\{-\frac{1}{r^2}\int_{r_2}^{r}Trdr+\frac{1}{r_3^2-r_1^2}\left[\int_{r_1}^{r_3}Trdr-\frac{r_3^2}{r^2}\int_{r_1}^{r_2}Trdr-\frac{r_1^2}{r^2}\int_{r_2}^{r_3}Trdr\right]+\frac{(r_2^2-r_1^2)(r^2-r_3^2)}{2(r_3^2-r_1^2)r^2}(T_1-T_0)\Big\}$$
$$\sigma_\theta=\frac{\alpha E}{1-\nu}\Big\{-T+\frac{1}{r^2}\int_{r_2}^{r}Trdr+\frac{1}{r_3^2-r_1^2}\left[\int_{r_1}^{r_3}Trdr+\frac{r_3^2}{r^2}\int_{r_1}^{r_2}Trdr+\frac{r_1^2}{r^2}\int_{r_2}^{r_3}Trdr\right]+\frac{(r_2^2-r_1^2)(r^2+r_3^2)}{2(r_3^2-r_1^2)r^2}(T_1-T_0)\Big\} \tag{21}$$

Substituting equation (20) into equation (14), the axial stress in the inner cylinder is,

$$\sigma_z=\frac{\alpha E}{1-\nu}\left[-T+\frac{2\nu}{(r_3^2-r_1^2)}\int_{r_1}^{r_3}Trdr-\frac{\nu(r_3^2-r_2^2)}{(r_3^2-r_1^2)}(T_1-T_0)\right] \tag{22}$$

This is the axial stress which is applied to keep the axial displacement w zero throughout. In the case of a long cylinder with unrestrained ends, a valid approximate solution except near the ends can be obtained by superposing simple tension or compression so as to reduce the resultant force on the ends, due to σ_z, to zero. The resultant of axial stress is,

$$\int_{r_1}^{r_2}\sigma_z\cdot 2\pi rdr=\frac{2\pi\alpha E}{1-\nu}\left[\frac{\nu(r^2-r_1^2)}{(r_3^2-r_1^2)}\int_{r_1}^{r_3}Trdr-\int_{r_1}^{r_2}Trdr-\frac{\nu(r_2^2-r_1^2)(r_3^2-r_2^2)}{2(r_3^2-r_1^2)}(T_1-T_0)\right] \tag{23}$$

and the resultant of constant axial stress C_3 is $C_3\pi(r_2^2-r_1^2)$. Hence the value of C_3 making the total axial force zero is given by

$$C_3=\frac{\alpha E}{(1-\nu)}\left[-\frac{2\nu}{(r_3^2-r_1^2)}\int_{r_1}^{r_3}Trdr+\frac{2}{(r_2^2-r_1^2)}\times\int_{r_1}^{r_2}Trdr+\frac{\nu(r_3^2-r_2^2)}{(r_3^2-r_1^2)}(T_1-T_0)\right] \tag{24}$$

Therefore the axial stress for generalized plane strain in inner cylinder is,

$$\sigma_z=\frac{\alpha E}{1-\nu}\left(-T+\frac{2}{r_2^2-r_1^2}\int_{r_1}^{r_2}Trdr\right) \tag{25}$$

By similar procedure, the axial stress in outer cylinder is,

$$\sigma_z=\frac{\alpha E}{1-\nu}\left(-T+\frac{2}{r_3^2-r_2^2}\int_{r_2}^{r_3}Trdr\right) \tag{26}$$

The strain components are given by equation (13) (17), (20), (21), (25) and (26).

Inner cylinder

$$\epsilon_r=\frac{\alpha}{1-\nu}\Big\{(1+\nu)T-\frac{(1+\nu)}{r^2}\int_{r_1}^{r}Trdr+\frac{(1-\nu)r^2-(1+\nu)r_1^2}{(r_3^2-r_1^2)r^2}\int_{r_1}^{r_3}Trdr-\frac{2\nu}{(r_2^2-r_1^2)}\times\int_{r_1}^{r_2}Trdr-\frac{(r_3^2-r_2^2)[(1-\nu)r^2-(1+\nu)r_1^2]}{2(r_3^2-r_1^2)r^2}\times(T_1-T_0)\Big\}$$

$$\epsilon_\theta=\frac{\alpha}{1-\nu}\Big\{\frac{(1+\nu)}{r^2}\int_{r_1}^{r}Trdr+\frac{(1-\nu)r^2+(1+\nu)r_1^2}{(r_3^2-r_1^2)r^2}\times\int_{r_1}^{r_3}Trdr-\frac{2\nu}{(r_2^2-r_1^2)}\int_{r_1}^{r_2}Trdr-\frac{(r_3^2-r_2^2)[(1-\nu)r^2+(1+\nu)r_1^2]}{2(r_3^2-r_1^2)r^2}\Big\}$$

$$\epsilon_z=\frac{\alpha}{1-\nu}\Big[\frac{2}{r_2^2-r_1^2}\int_{r_1}^{r_2}Trdr-\frac{2\nu}{(r_3^2-r_1^2)}\times\int_{r_1}^{r_3}Trdr+\frac{\nu(r_3^2-r_2^2)}{(r_3^2-r_1^2)}(T_1-T_0)\Big] \tag{27}$$

$=$constant

Outer cylinder

$$\epsilon_r=\frac{\alpha}{1-\nu}\Big\{(1+\nu)T-\frac{(1+\nu)}{r^2}\int_{r_2}^{r}Trdr+\frac{(1-\nu)}{r_3^2-r_1^2}\times\int_{r_1}^{r_3}Trdr-\frac{(1+\nu)r_3^2}{(r_3^2-r_1^2)r^2}\int_{r_1}^{r_2}Trdr-\Big[\frac{(1+\nu)r_1^2}{(r_3^2-r_1^2)r^2}+\frac{2\nu}{(r_3^2-r_2^2)}\Big]\int_{r_2}^{r_3}Trdr+\frac{(r_2^2-r_1^2)[(1-\nu)r^2-(1+\nu)r_3^2]}{2(r_3^2-r_1^2)r^2}(T_1-T_0)-(1-\nu)(T_1-T_0)\Big\}$$

$$\epsilon_\theta=\frac{\alpha}{1-\nu}\Big\{\frac{(1+\nu)}{r^2}\int_{r_2}^{r}Trdr+\frac{1-\nu}{r_3^2-r_1^2}\int_{r_1}^{r_3}Trdr+\frac{(1+\nu)r_3^2}{(r_3^2-r_1^2)r^2}\int_{r_1}^{r_2}Trdr+\Big[\frac{(1+\nu)r_1^2}{(r_3^2-r_1^2)r^2}+\frac{2\nu}{r_3^2-r_1^2}\Big]\int_{r_2}^{r_3}Trdr+\frac{(r_2^2-r_1^2)[(1-\nu)r^2+(1+\nu)r_3^2]}{2(r_3^2-r_1^2)r^2}(T_1-T_0)-(1-\nu)(T_1-T_0)\Big\}$$

$$\epsilon_z=\frac{\alpha}{1-\nu}\Big[\frac{2}{r_3^2-r_2^2}\int_{r_2}^{r_3}Trdr-\frac{2\nu}{(r_3^2-r_1^2)}\times\int_{r_1}^{r_3}Trdr-\frac{\nu(r_2^2-r_1^2)}{(r_3^2-r_1^2)}\Big] \tag{28}$$

$=$constant

2.2.2 Plane Stress Problem

In this case, the stresses are specified only by σ_r and σ_θ, and independent of axial distance, i.e, $\sigma_z=0$. The stress-strain relations are,

$$\epsilon_r=\frac{1}{E}(\sigma_r-\nu\sigma_\theta)+\alpha(T-T_0)$$

$$\epsilon_\theta=\frac{1}{E}(\sigma_\theta-\nu\sigma_r)+\alpha(T-T_0) \tag{29}$$

By similar procedure with the case of plane strain problem, the stress and strain components are determined.

Inner cylinder

$$\sigma_r=\alpha E\Big[-\frac{1}{r^2}\int_{r_1}^{r}Trdr+\frac{r^2-r_1^2}{(r_3^2-r_1^2)r^2}\int_{r_1}^{r_3}Trdr-\frac{(r_3^2-r_2^2)(r^2-r_1^2)}{2(r_3^2-r_1^2)r^2}(T_1-T_0)\Big]$$

$$\sigma_\theta=\alpha E\Big[-T+\frac{1}{r^2}\int_{r_1}^{r}Trdr+\frac{r^2+r_1^2}{(r_3^2-r_1^2)}\int_{r_1}^{r_3}Trdr-\frac{(r_3^2-r_2^2)(r^2+r_1^2)}{2(r_3^2-r_1^2)r^2}(T_1-T_0)\Big] \tag{30}$$

Outer cylinder

$$\sigma_r=\alpha E\Big\{-\frac{1}{r^2}\int_{r_2}^{r}Trdr+\frac{1}{r_3^2-r_1^2}\Big[\int_{r_1}^{r_3}Trdr-\frac{r_3^2}{r^2}\int_{r_1}^{r_2}Trdr-\frac{r_1^2}{r^2}\int_{r_2}^{r_3}Trdr\Big]+\frac{(r_2^2-r_1^2)(r^2-r_3^2)}{2(r_3^2-r_1^2)r^2}(T_1-T_0)\Big\}$$

$$\sigma_\theta=\alpha E\Big\{-T+\frac{1}{r^2}\int_{r_2}^{r}Trdr+\frac{1}{r_3^2-r_1^2}\Big[\int_{r_1}^{r_3}Trdr+\frac{r_3^2}{r^2}\int_{r_1}^{r_2}Trdr+\frac{r_1^2}{r^2}\int_{r_2}^{r_3}Trdr\Big]+\frac{(r_2^2-r_1^2)(r^2+r_3^2)}{2(r_3^2-r_1^2)r^2}(T_1-T_0)\Big\} \tag{31}$$

Inner cylinder

$$\epsilon_r=\alpha\Big\{(1+\nu)T-\frac{1+\nu}{r^2}\int_{r_1}^{r}Trdr+\frac{[(1-\nu)r^2-(1+\nu)r_1^2]}{(r_3^2-r_1^2)r^2}\Big[\int_{r_1}^{r_3}Trdr-\frac{(r_3^2-r_2^2)}{2}(T_1-T_0)\Big]\Big\}$$

$$\epsilon_\theta=\alpha\Big\{\frac{1+\nu}{r^2}\int_{r_1}^{r}Trdr+\frac{[(1-\nu)r^2+(1+\nu)r_1^2]}{(r_3^2-r_1^2)r^2}\Big[\int_{r_1}^{r_3}Trdr-\frac{r_3^2-r_2^2}{2}(T_1-T_0)\Big]\Big\} \tag{32}$$

Outer cylinder

$$\epsilon_r=\alpha\Big\{(1+\nu)T-\frac{(1+\nu)}{r^2}\int_{r_2}^{r}Trdr+\frac{1}{r_3^2-r_1^2}\Big[(1-\nu)\int_{r_1}^{r_3}Trdr-\frac{(1+\nu)r_3^2}{r^2}\int_{r_1}^{r_2}Trdr-\frac{(1+\nu)r_1^2}{r^2}\int_{r_2}^{r_3}Trdr\Big]+\frac{(r_2^2-r_1^2)[(1-\nu)r^2-(1+\nu)r_3^2]}{2(r_3^2-r_1^2)r^2}(T_1-T_0)-(T_1-T_0)\Big\}$$

$$\epsilon_\theta=\alpha\Big[\frac{(1+\nu)}{r^2}\int_{r_2}^{r}Trdr+\frac{1}{r_3^2-r_1^2}\Big[(1-\nu)\int_{r_1}^{r_3}Trdr+\frac{(1+\nu)r_3^2}{r^2}\int_{r_1}^{r_2}Trdr+\frac{(1+\nu)r_1^2}{r^2}\int_{r_2}^{r_3}Trdr\Big]+\frac{(r_2^2-r_1^2)[(1-\nu)r^2+(1+\nu)r_3^2]}{2(r_3^2-r_1^2)r^2}(T_1-T_0)-(T_1-T_0)\Big] \quad (33)$$

Comparing equation (20), (21) with the equation (30) and (31), the radial and tangential stresses in plane stress problem are $(1-\nu)$ times those in plane strain problem.

3. Numerical Results and Discussion

3.1 Input Data

Computation was performed for AISI 1040 carbon steel which is generally used for crankshafts, axies and connecting rods, using IBM 1130 in Seoul National University.

Table 1. Material Properties

a. Chemical Composition

AISI	C	Mn	P	S
C-1040	0.37-0.44	0.60-0.90	0.04max.	0.05max.

b. Material Properties

Modulus of elasticity	E	3.0×10^7	lb/in²
Density	ρ	489	lb/ft³
Specific heat	Cp	30.111	Btu/lb-°F
Thermal conductivity	k	31	Btu/hr-ft-°F
Thermal diffusivity	β	0.570	ft²/hr
Heat transfer coefficient**	h	4.83	Btu/hr-ft²-°F
Poisson's ratio	ν	0.3	
Thermal linear expansion coefficient	α	6.33×10^{-6}	ft/ft-°F
Tensile strength		125,000-98,000	lb/in²
Yield strength		104,000-60,000	lb/in²

** Heat transfer coefficient for radiation and convection within 500 °F temperature difference when the room temperature is 80°F.

c. Dimensions

$r = 3.0$ inches

$r = 3.5$ inches

$r = 4.0$ inches

The initial inner radius of outer cylinder is made $\alpha(T_1-T_0)$ less than the outer radius of inner cylinder.

In case of low carbon steel, the variation of material properties are not significant as shown in Fig. 3, Fig. 4, and Fig. 5, if the range of working temperature is within 500 °F.

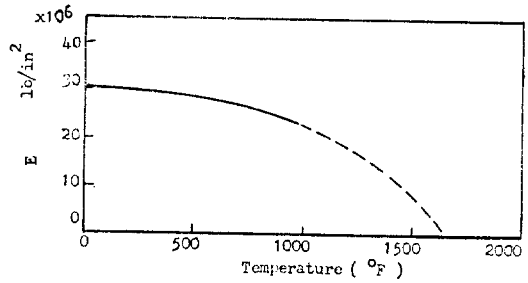

Fig. 3. Variation of Modulus of Elasticity

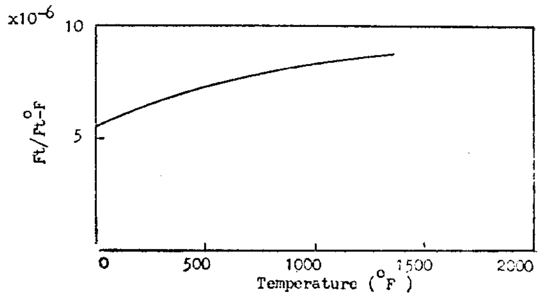

Fig. 4 Variation of Thermal Linear Expansion Coefficient

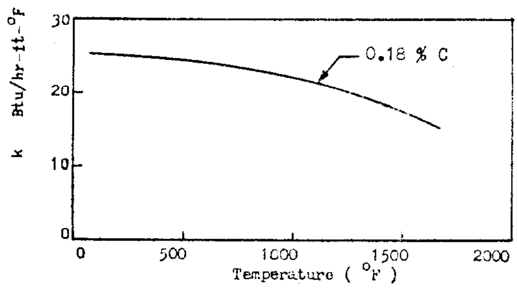

Fig. 5. Variation of Thermal Conductivity

Chemical composition of steel for Fig. 3, Fig. 4 and Fig. 5 is C-0.20%, Mn-0.6%, P-0.011% and S-0.025%. [4]

In computation of the infinite series summation. the first 30 terms are considered, for the rest do not have significant order in engineering sence. Stresses and strains are computed only in plane strain problem, for the stresses in plane stress problem are $(1-\nu)$ times those in plane strain problem. The values of temperature, stresses and strains are computed at 0.1, 0.2, 0.5, 1.0, 2.0, 3.0, 5.0, 10.0, 30.0, 60.0, 90.0, 600.0, 1800.0, and 3600.0 seconds after fitting.

3.2 Temperature Distribution

The transient temperature distribution is shown in Fig. 6. The uniform temperature distribution is achieved approximately 60 seconds after fitting. The reason why the uniform temperature is achieved at the higher temperature than $\frac{T_1-T_0}{2}$ is that the total heat capacity of the inner cylinder is smaller than that of outer cylinder and the heat loss due to convection and radiation effect is not significant.

$$\frac{dT}{dr}=-\frac{h}{k}(T_1-T_0)$$
$$=-0.013(T_1-T_0)\ ^\circ\mathrm{F/in}$$

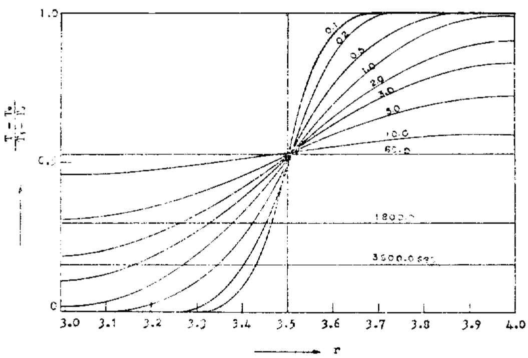

Fig. 6. Temperature Distribution

The temperature gradient at the outer surface and Biot's number calculated in equation(34) show that internal conduction resistance is negligible in comparison with surface convection and radiation resistance. Since no change of thermal stress is happened after uniform temperature distribution is achieved, for this problem the temperature distribution is affected only by the conduction effect. Also the assumption that the inner surface of inner cylinder is insulated can be supposed not to have a significant effect on the solution.

If the inner cylinder has no concentric hole, that is $r_1=0$, then $R_1=0$, $R_3=1$ in equation (11). Thus

$$T(r,t)=T_0+(T_1-T_0)\sum_{j=1}^{\infty}\frac{2}{\lambda_j}$$
$$\times\frac{J_1(\lambda_j)-\frac{r_2}{r_3-r_1}J_1\left(\lambda_j\frac{r_2}{r_3-r_1}\right)}{[J_1(\lambda_j)]^2+[J_0(\lambda_j)]^2}\times J_0\left(\lambda_j\times\frac{r}{r_3-r_1}\right)$$
$$\times\exp\left[-\lambda_j^2\frac{\beta}{(r_3-r_1)^2}\,t\right] \tag{35}$$

where λ_j's are solutions of

$$\lambda J_1(\lambda)-\mu J_0(\lambda)=0 \tag{36}$$

If the temperature distribution of a single solid cylinder is required, then $r_1=r_2=0$, that is $R_1=R_2=0$, $R_3=1$. Thus,

$$T(r,t)=T_0+(T_1-T_0)\sum_{j=1}^{\infty}\frac{2}{\lambda_j}$$
$$\times\frac{J_1(\lambda_j)}{[J_1(\lambda_j)]^2+[J_0(\lambda_j)]^2}\times J_0\left(\lambda_j\frac{r}{r_3-r_1}\right),$$
$$\times\exp\left[-\lambda_j^2\frac{\beta}{(r_3-r_1)^2}\,t\right] \tag{37}$$

3.3 Thermal Stresses and Strains

The transient thermal stress distributions are shown in Fig. 7, Fig. 8 and Fig. 9. The radial stress is always compressive throughout the cylinder and has the maximum value at the interfering surface, which contributes to the holding power of shrink-fitting, at the interfering boundary in Fig. 7. The tangential stress is compressive in inner cylinder and tensile in outer cylinder, for the inner cylinder is restrained from expansion and the outer cylinder from contraction in Fig. 8. It is also shown in Fig. 8 that at the interfering boundary in inner cylinder and outer cylinder, the magnitude of tangential stress has almost the same value, though the temperature decreases until the uniform distribution is achieved. The maximum stress is produced at the inner surface of the inner cylinder and outer cylinder. The axial stress has the maximum value at the interfering boundary immediately after fitting and decreases to almost zero when the uniform temperature distribution is achieved. Actually, it has a negligible residual stress as shown in Fig. 9. It takes about 60 seconds to achieve the residual stresses and uniform temperature distribution does not change the stress distribution afterwards, even though the temperature level is dropped.

The transient thermal strains are shown in Fig. 10 and Fig. 11. It is shown in Fig. 11 that the inner cylinder is expanded and the outer cylinder is contracted due to the heat transfer between both cylinder. The maximum radial strain is produced at the intefering boundary where the large temperature variation is happened. The signs of tangential strain in Fig. 11 in both cylinder are opposite to those of radial strain respectively for the Poisson's effect. The axial strain is apparently constant in generalized plane strain problem. The axial strainin inner cylinder is $(T_1-T_0)\times 4.8246\times 10^{-6}$ and in outer cylinder $(T_1-T_0)\times 2.1081\times 10^{-6}$ and these are not plottet.

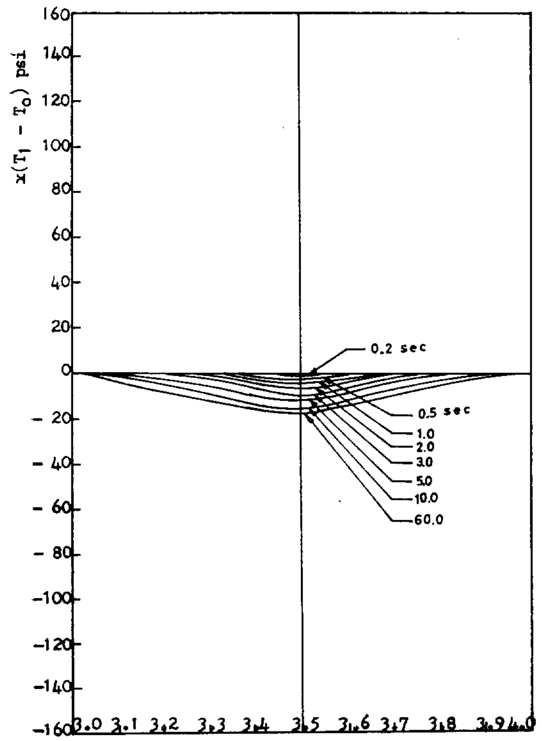

Fig. 7. Radial Stress Distriaution

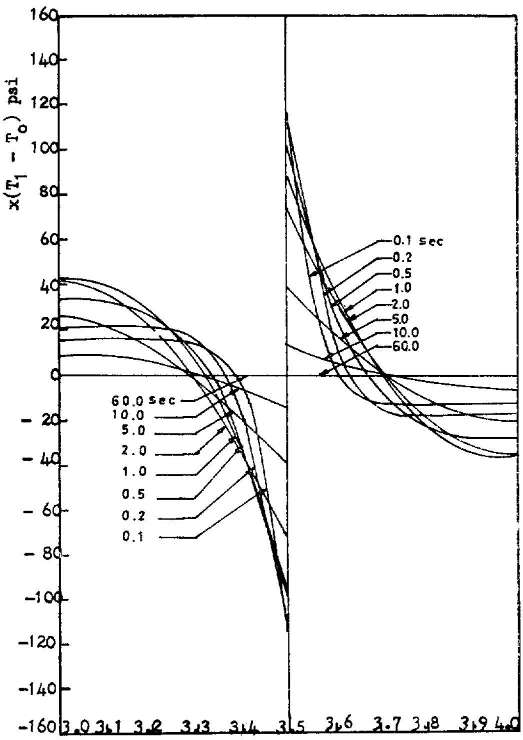

Fig. 9. Axial Stress Distribution

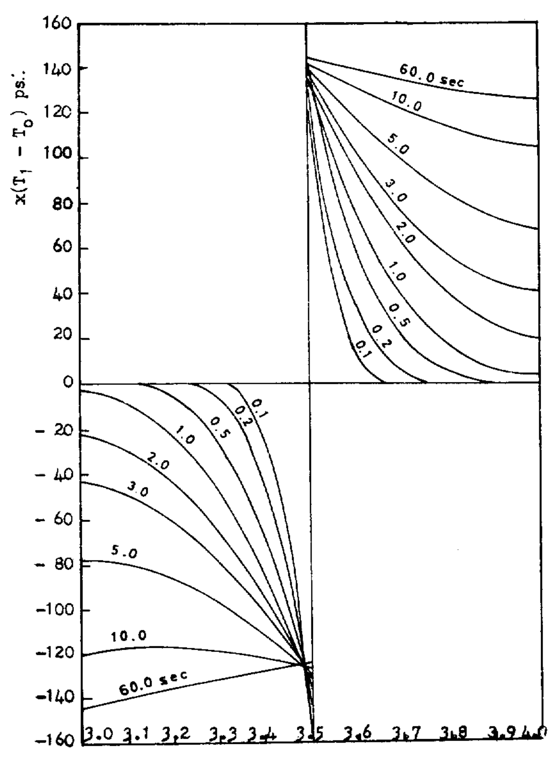

Fig. 8. Tangential Stress Distribution

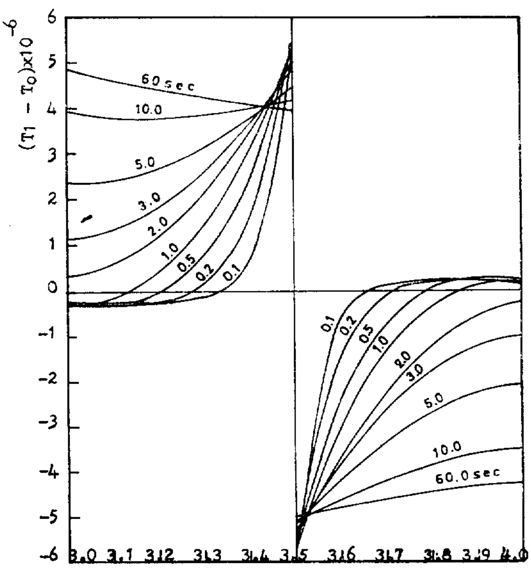

Fig. 10. Radial Strain

The results of the work reported here can be applied to the problem of cylinders which are subject to the additional inner pressure as gun firing pressure with slight change of boundary conditions. And the problem of steel tire under operation can also be studied with adding the centrifugal forces

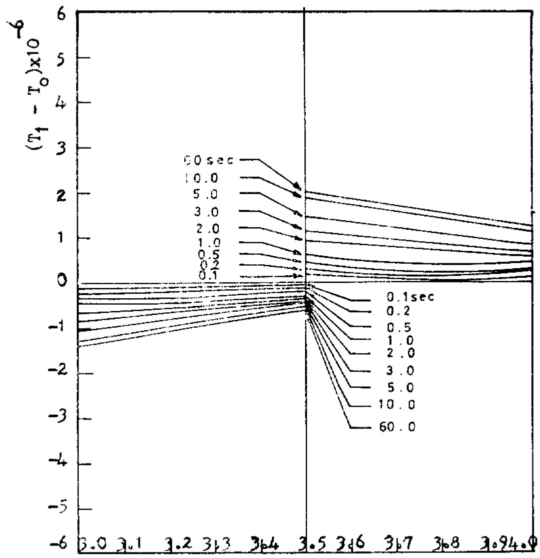

Fig. 11. Tangential Strain

in the compatibility equation.

If the inner cylinder has no concentric hole, corresponding to the temperature distribution equation (35), the constant B_1 in equation (15) must be zero for the stresses are finite in the center of the inner cylinder. The other three constants are determined with the rest three boundary conditions.

The radial and tangential stresses, if the inner cylinder is solid, are as following,

Inner cylinder

$$\sigma_r = -\frac{\alpha E}{(1-\nu)}\frac{1}{r^2}\int_0^r Trdr + \frac{\alpha E}{(1-\nu)r_3^2}\int_0^{r_3} Trdr - \frac{\alpha E}{2(1-\nu)}\left(1-\frac{r_2^2}{r_3^2}\right)(T_1 - T_0)$$

$$\sigma_\theta = -\frac{\alpha ET}{1-\nu} + \frac{\alpha E}{(1-\nu)r^2}\int_0^r Trdr + \frac{\alpha E}{(1-\nu)r_3^2}\times \int_0^{r3} Trdr - \frac{\alpha E}{2(1-\nu)}\left(1-\frac{r_2^2}{r_3^2}\right)(T_1 - T_0) \tag{38}$$

Outer cylinder

$$\sigma_r = \frac{-\alpha E}{(1-\nu)r^2}\left\{\int_{r_2}^r Trdr + \int_0^{r_2} Trdr\right\} + \frac{\alpha E}{(1-\nu)r_3^2}\int_0^{r3} Trdr + \frac{\alpha E r_2^2}{2(1-\nu)}\left(\frac{1}{r_3^2} - \frac{1}{r_2^2}\right)\times (T_1 - T_0)$$

$$\sigma_\theta = -\frac{\alpha ET}{1-\nu} + \frac{\alpha E}{(1-\nu)r^2}\left[\int_{r_2}^r Trdr - \int_0^{r_2} Trdr\right] + \frac{\alpha E}{(1-\nu)r_3^2}\int_0^{r3} Trdr + \frac{\alpha E r_2^2}{2(1-\nu)}\left(\frac{1}{r_3^2} + \frac{1}{r^2}\right)\times (T_1 - T_0) \tag{39}$$

Temperature in equation (38) and (39) can be found in equation (35).

3.4 Camparison of the Theoretical Results and the Values from Lamé's Formula

Lamé's formulae of thick cylinder give the residual stresses from the relation between shrinking pressure and radial interference in plane stress problem.[10]

Inner cylinder

$$\sigma_r = P_s\left[\frac{r_1^2 r_2^2}{(r_2^2 - r_1^2)r^2} - \frac{r_2^2}{(r_2^2 - r_1^2)}\right]$$

$$\sigma_\theta = -P_s\left[-\frac{r_1^2 r_2^2}{(r_2^2 - r_1^2)r^2} + \frac{r_2^2}{(r_2^2 - r_1^2)}\right] \tag{40}$$

Outer cylinder

$$\sigma_r = P_s\left[\frac{-r_2^2 r_3^2}{(r_3^2 - r_2^2)r^2} + \frac{r_2^2}{(r_3^2 - r_2^2)}\right]$$

$$\sigma_\theta = P_s\left[\frac{r_2^2 r_3^3}{(r_3^2 - r_2^2)r^2} + \frac{r_2^2}{(r_3^2 - r_2^2)}\right] \tag{41}$$

where shrinking pressure is

$$P_s = \frac{\delta E}{2r_2^3}\times\frac{(r_3^2 - r_2^2)(r_2^2 - r_1^2)}{r_3^2 - r_1^2} \tag{42}$$

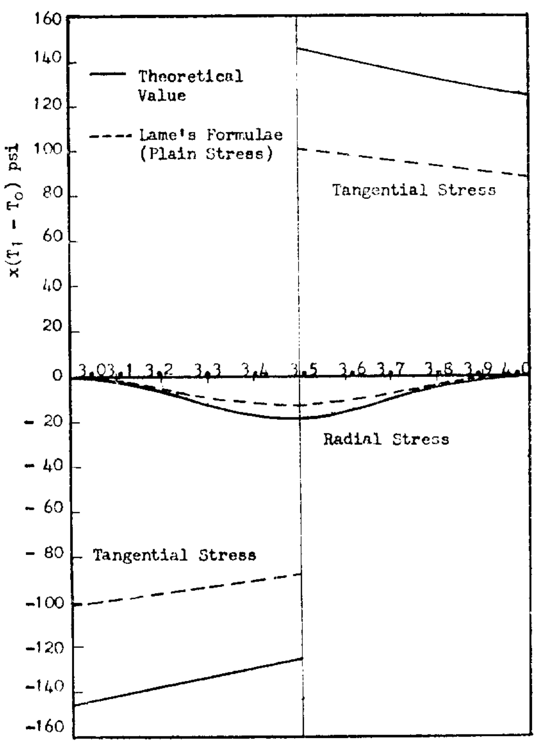

Fig. 12. Comparison of Theoretical Residual Stresses and the Stresses of Lame's Formulae

and δ is the radial tolerance between the outer radius of inner cylinder and inner radius of outer cylinder. The comparison of the theoretical value of residual stresses in plane strain problem and the value from Lamé's formula is shown in Fig.12. Fig. 12 shows a good coincidence of stress distribution tendency but the magnitudes of theoretical value is about 1.4 times that of Lamé's formulae, for Lamé's formulae are analyzed in plane stress problem.

4. Conclusion

Analytical solutions for transient temperature distribution, thermal stresses and strains in a composite cylinder due to shrink fitting is obtained. From the sample calculation, the following conclusion can be derived.

1) Temperature distribution can be determined by considering conduction effect only for low Biot's number.
2) The magnitude of tangential stress at the interfering surface keeps almost same value during the uniform temperature is achieved.
3) The maximum radial stress is produced at intefering surface and tangential stress at the inner surface of each cylinder.
4) The residual stress is achieved when the uniform temperature distribution is achieved.

Comparison of analytic solution and Lamé's thick cylinder theory in residual stresses shows accurate coincidence.

5. Acknowledgement

The authors would like to express their gratitude to the professors of Engineering College of S.N.U. for their helpful discussion and continuous encouragement.

6. References

1) Boelter, Cherry, Johnson and Martinelli, "Heat Transfer", V-14, *University of California Press* Berkeley, California, 1946.
2) Chieh-Chien Chang and Wen-Hwa Chu, "Stresses in a Metal Tube under Both High Radial Temperature Variation and Internal Pressure" *Journal of Applied Mechanics*, 1954, pp.101-108.
3) Tu-Lung Weng, "Thermal Stresses in Anisotropic Hollow Cylinders", *Journal of Basic Engineering* 1965, pp.391-397.
4) Lambert Tall, "Residual Stresses in Welded Plate—A Theoretical Study" *Welding Journal R.Supp.*, January 1964.
5) R.J. Grosh and E.A. Trabant, "Arc Welding Temperature", *Welding Journal R. Supp.* August 1956.
6) Zae Geun Kim, Hyochul Kim, "On the Thermal Stress and Residual Stress Distributions in a Aluminum Alloy Plate due to Resistance Spot Welding", *Journal of SNAK*, 1972, Vol. 9, No. 2.
7) Shangjun Lim, "The Stress Distribution around a Reinforces Circular Hole in a Plate under General Biaxial Loading", *Journal of SNAK* 1969, Vol. 6 No.1.
8) B.E. Gatwood, "*Thermal Stresses*", McGraw-Hill Book Company, 1957.
9) S.P. Timoshenko and J.N. Goodier, "*Theory of Elasticity*" third edition, McGraw-Hill Book Company, 1970.
10) Fred B. Seely, James O. Smith, "*Advanced Mechanics of Meterials*" second edition, John Wiley and Sons, Inc. 1952 pp.295-341.
11) J.P. Holman, "*Heat Transfer*" third edition, McGraw-Hill Book Company, 1972.