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ON AN INTEGRAL TRANSFORM INVOLVING MEIJER'S G-FUNCTIONS

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1. A Fourier-type integral transform has been introduced [3, p. 298] whose kernel is

$$\sqrt{2}\,G^{q,p}_{2p,2q}\left[x^2/4\left|\begin{matrix}\alpha_1,\cdots,\alpha_p,\frac{1}{2}-\alpha_1,\cdots,\frac{1}{2}-\alpha_p\\ \beta_1,\cdots,\beta_q,\frac{1}{2}-\beta_1,\cdots,\frac{1}{2}-\beta_q\end{matrix}\right.\right].$$

Here $G^{m,n}_{p,q}$ denotes the Meijer G-function [7, p. 143]. Following the same analysis as in [3], one can easily show that the function

$$k(x)=\gamma\mu^{\gamma/2}x^{(\gamma-1)/2}G^{q,p}_{2p,2q}\left[(\mu x)^{\gamma}\left|\begin{matrix}a_1,\cdots,a_p,-a_1,\cdots,-a_p\\ b_1,\cdots,b_q,-b_1,\cdots,-b_q\end{matrix}\right.\right] \tag{1.1}$$

also plays the role of a Fourier kernel. Here μ and γ are positive real constants, p and q positive integers such that

$$q-1\geq p\geq 0 \tag{1.2}$$

and a_j, $j=1,\cdots,p$ and b_h, $h=1,\cdots,q$, are numbers satisfying

$$\begin{aligned}&a_j-b_h\neq 1,2,3,\cdots,\ j=1,\cdots,p,\ h=1,\cdots,q;\\&b_j-b_h\neq 0,\pm1,\pm2,\cdots,\ j=1,\cdots,q,\ h=1,\cdots,q,\ j\neq h;\\&\mathrm{Re}\left(\frac{1}{2}-a_j\right)>0,\ j=1,\cdots,p;\\&\mathrm{Re}\left(\frac{1}{2}+b_h\right)>0,\ h=1,\cdots,q.\end{aligned} \tag{1.3}$$

The conditions (1.3) ensure that the poles of $\Gamma(b_j-s)$ and $\Gamma(1-a_j+s)$ lie on the opposite sides of the contour used in the definition of the G-function and that they are simple poles.

In obtaining the kernel (1.1), Mellin transform of $k(x)$ has been used. The convergence of the Mellin integral involved is ensured under the assumption (1.2)

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and the fact that arg $\mu=0$. See [7, p. 159, Case 3].

The integral transform we discuss in this note arises from the integral formula

(1.4) $$f(x)=\int_0^\infty k(xu)\,du\int_0^\infty k(ut)f(t)\,dt,$$

where $k(x)$ is an in (1.1). This formula gives rise to the reciprocal relations

(1.5) $$F(x)=\int_0^\infty k(xu)f(u)\,du,$$

(1.6) $$f(x)=\int_0^\infty k(xu)F(u)\,du,$$

connecting two functions $f(x)$ and $F(x)$. We will call each of the two functions so related the *G-transform of each other*. If, further, $F(x)=f(x)$ so that

(1.7) $$f(x)=\int_0^\infty k(xu)f(u)\,du,$$

then $f(x)$ is called *self-reciprocal in the G-transform.*

The kernel (1.1) is a very general one. It contains as its particular cases Fourier-type kernels studied by various authors from time to time. Some of them were listed in [4, pp. 957–958].

The formula (1.4), with the left-hand side replaced by $\{f(x+0)+f(x-0)\}/2$ at the points of discontinuity of $f(x)$, has been proved [1, p. 400] under the hypothesis that $t^\sigma f(t)\in L(0,\infty)$, with suitable σ, and that $f(t)$ is of bounded variation in the neighborhood of the point $t=x$. See [5, p. 28] also. However, when we come to study the relations (1.5), (1.6) directly, we find that the theory of ordinary convergence is not always enough. A case in which a satisfactory theory can be developed is that in which $f\in L_2(0,\infty)$; even in this case, the integrals in (1.5), (1.6) do not generally exist, and we have to express the reciprocal relations in the form

(1.8) $$F(x)=\frac{d}{dx}\int_0^\infty \frac{k_1(xu)}{u}f(u)\,du,$$

(1.9) $$f(x)=\frac{d}{dx}\int_0^\infty \frac{k_1(xu)}{u}F(u)\,du,$$

where $k_1(x)=\int_0^x k(u)du$. In case we can differentiate with respect to x under the integral sign, the formulas (1.8), (1.9) reduce to (1.5) and (1.6).

In [6, p. 96] Kesarwani obtained a necessary condition, given below as Theorem 1, so that a pair of functions $f(x)$, $F(x)$ are G-transforms of each other. The purpose of this note is to establish that the condition is sufficient also.

2. In [6] Kesarwani shows that the function

$$x^{(\gamma-1)/2}G^{q,p}_{p,q}\left[(\sqrt{\mu}x)^{\gamma}\middle|\begin{matrix}a_1, \cdots, a_p\\ b_1, \cdots, b_q\end{matrix}\right]$$

is self-reciprocal in the G-transform under the conditions (1.3) and belongs to L_2 $(0, \infty)$. By making obvious changes of variables, he also shows that if $y>0$ the functions

$$(2.1)\qquad \begin{gathered}(yx)^{(\gamma-1)/2}G^{q,p}_{p,q}\left[(\sqrt{\mu}yx)^{\gamma}\middle|\begin{matrix}a_1, \cdots, a_p\\ b_1, \cdots, b_q\end{matrix}\right]\\ y^{-1}(x/y)^{(\gamma-1)/2}G^{q,p}_{p,q}\left[(\sqrt{\mu}x/y)^{\gamma}\middle|\begin{matrix}a_1, \cdots, a_p\\ b_1, \cdots, b_q\end{matrix}\right]\end{gathered}$$

considered as functions of x form a pair of G-transforms.

THEOREM 1. [6, p. 96]. *Let* (i) $\gamma>0$, $\mu>0$, (ii) $\mathrm{Re}\left(\frac{1}{2}-a_j\right)>0$, $j=1, \cdots, p$, (iii) $\mathrm{Re}\left(\frac{1}{2}+b_j\right)>0$, $j=1, \cdots, q$, (iv) $q-1\geq p\geq 0$. *A necessary condition that functions $f(x)$ and $F(x)$ in $L_2(0, \infty)$ be a pair of G-transforms is that*

$$(2.2)\qquad \begin{aligned}&\int_0^\infty f(t)\,(xt)^{(\gamma-1)/2}\,G^{q,p}_{p,q}\left[(\sqrt{\mu}xt)^{\gamma}\middle|\begin{matrix}a_1, \cdots, a_p\\ b_1, \cdots, b_q\end{matrix}\right]dt\\ &=x^{-1}\int_0^\infty F(t)(t/x)^{(\gamma-1)/2}\,G^{q,p}_{p,q}\left[(\sqrt{\mu}t/x)^{\gamma}\middle|\begin{matrix}a_1, \cdots, a_p\\ b_1, \cdots, b_q\end{matrix}\right]dt\end{aligned}$$

for all $x>0$.

3. The Mellin transform of $f(x)$ is denoted by $\mathfrak{M}\{f(x)\}$. If $\mathfrak{M}\{f(x)\}=F(s)$ we shall also write $f(x)=\mathfrak{M}^{-1}\{F(s)\}$ and $\mathfrak{M}^{-1}$ will denote the inverse Mellin transform. If $f(x)\in L_2(0, \infty)$ and the l. i. m. is with index 2 then

$$(3.1)\qquad F(s)=\mathfrak{M}\{f(x)\}=\underset{N\to\infty}{\mathrm{l.\,i.\,m.}}\int_{1/N}^{N} f(x)\,x^{s-1}\,dx$$

and also $F(s)\in L_2\left(\frac{1}{2}-i\infty,\ \frac{1}{2}+i\infty\right)$. If $F(s)\in L_2\left(\frac{1}{2}-i\infty,\ \frac{1}{2}+i\infty\right)$ then

$$f(x)=\mathfrak{M}^{-1}\{F(s)\}=\frac{1}{2\pi i}\underset{N\to\infty}{\mathrm{l.i.m.}}\int_{\frac{1}{2}-iN}^{\frac{1}{2}+iN} F(s)\,x^{-s}\,ds \tag{3.2}$$

and also $f(x)\in L_2(0,\ \infty)$. See Theorem 71, [8, p.94].

The following result of Fox [2, p.458] will be useful in our work.

THEOREM 2. *If* (i) $x>0$, (ii) $f(z)$ *and* $g(z)$ *both belong to* $L_2(0,\infty)$, (iii) $\mathfrak{M}\{f(z)\}=F(s)$, $\mathfrak{M}\{g(z)\}=G(s)$ *and* $G(s)$ *is bounded on the line* $s=\frac{1}{2}+it$, $-\infty<t<\infty$, *then*

$$\int_0^\infty g(xz)\,f(z)\,dz\in L_2(0,\ \infty)\ \textit{and} \tag{3.3}$$

$$\mathfrak{M}\{\int_0^\infty g(xz)\,f(z)\,dz\}=G(s)\,F(1-s), \tag{3.4}$$

where the integrals of (3.3) *and* (3.4) *are taken as functions of* x.

THEOREM 3. *If* (i) $x>0$, $\mu>0$, $\gamma>0$, (ii) $\mathrm{Re}\left(\frac{1}{2}-a_j\right)>0$, $j=1,\ \cdots,\ p$, (iii) $\mathrm{Re}\left(\frac{1}{2}+b_j\right)>0$, $j=1,\ \cdots,\ q$, (iv) $q-1\geq p\geq 0$, (v) $f(x)\in L_2(0,\ \infty)$ *and* (vi)

$$\int_0^\infty f(t)(xt)^{(\gamma-1)/2}\,G^{q,p}_{p,q}\left[(\sqrt{\mu}xt)^\gamma\,\middle|\,\begin{matrix}a_1,\ \cdots,\ a_p\\ b_1,\ \cdots,\ b_q\end{matrix}\right]dt=0 \tag{3.5}$$

then $f(x)=0$ *almost everywhere.*

PROOF. Let

$$\begin{aligned} h(x)&=x^{(\gamma-1)/2}G^{q,p}_{p,q}\left[(\sqrt{\mu}x)^\gamma\,\middle|\,\begin{matrix}a_1,\ \cdots,\ a_p\\ b_1,\ \cdots,\ b_q\end{matrix}\right]\\ &=\mu^{(1-\gamma)/4}\,G^{q,p}_{p,q}\left[(\sqrt{\mu}x)^\gamma\,\middle|\,\begin{matrix}a'_1,\ \cdots,\ a'_p\\ b'_1,\ \cdots,\ b'_q\end{matrix}\right]\end{aligned} \tag{3.6}$$

where

$$\begin{aligned} a'_j&=a_j+\frac{1}{2}-\frac{1}{2\gamma},\quad j=1,\ \cdots,\ p,\\ b'_j&=b_j+\frac{1}{2}-\frac{1}{2\gamma},\quad j=1,\ \cdots,\ q.\end{aligned} \tag{3.7}$$

Since $h(x)\in L_2(0,\ \infty)$, see [7, §5.6.2], we may obtain its Mellin transform as

(3.8) $$H(s)=\mathfrak{M}\{h(x)\}=\frac{1}{\gamma}\mu^{(1-\gamma)/4-\frac{s}{2}}\prod_{j=1}^{q}\Gamma\left(b_j'+\frac{s}{\gamma}\right)\prod_{j=1}^{p}\Gamma\left(1-a_j'-\frac{s}{\gamma}\right),$$

with $H(S)$ belonging to $L_2\left(\frac{1}{2}-i\infty,\ \frac{1}{2}+i\infty\right)$. Hence

(3.9) $$H\left(\frac{1}{2}+it\right)=\frac{1}{\gamma}\mu^{-\frac{\gamma}{4}-\frac{t}{2}i}\prod_{j=1}^{q}\Gamma\left(\frac{1}{2}+b_j+i\frac{t}{\gamma}\right)\prod_{j=1}^{p}\Gamma\left(\frac{1}{2}-a_j-i\frac{t}{\gamma}\right).$$
$$-\infty<t<\infty.$$

Using the asymptotic expansion of gamma function [7, p. 33]

(3.10) $$|\Gamma(x+iy)|\sim\sqrt{2\pi}\,|y|^{x-\frac{1}{2}}e^{-\frac{\pi}{2}|y|},\quad x,\ y \text{ real},\ |x| \text{ finite},\ |y|\to\infty,$$

we find that

(3.11) $$\left|H\left(\frac{1}{2}+it\right)\right|\sim A|t|^B e^{-\frac{\pi}{2}(q+p)|t|/\gamma},\quad |t|\to\infty$$

where A, B are some constants which are independent of $|t|$. Therefore we know that $H(s)$ is bounded on the line $s=\frac{1}{2}+it$, $-\infty<t<\infty$. Hence by Theorem 2

$$\int_0^\infty k(xt)f(t)dt\in L_2(0,\ \infty)$$

and

(3.12) $$0=\mathfrak{M}\{\int_0^\infty h(xt)f(t)\,dt\}=H(s)\,F(1-s)$$

$$=\frac{1}{\gamma}\mu^{\frac{1-\gamma}{4}-\frac{s}{2}}\prod_{j=1}^{q}\Gamma\left(\frac{1}{2}-\frac{1}{2\gamma}+b_j+\frac{s}{\gamma}\right)\prod_{j=1}^{p}\Gamma\left(\frac{1}{2}+\frac{1}{2\gamma}-a_j-\frac{s}{\gamma}\right)F(1-s).$$

Since $\Gamma(x)$ is never zero, we must have $F(1-s)=0$, i.e. $F(s)=0$, hence $f(x)=0$ almost everywhere. This completes the proof of the theorem.

THEOREM 4. *Let* (i) $\gamma>0$, $x>0$, $\mu>0$, (ii) $\mathrm{Re}\left(\frac{1}{2}-a_j\right)>0$, $j=1,\ \cdots,\ p$, (iii) $\mathrm{Re}\left(\frac{1}{2}+b_j\right)>0$, $j=1,\ \cdots,\ q$, (iv) $q-1\geq p\geq 0$. *A sufficient condition that the functions* $f(x)$ *and* $F(x)$ *in* $L_2(0,\infty)$ *be a pair of G-transforms is that*

(3.13) $$\int_0^\infty f(t)(xt)^{(\gamma-1)/2}G_{p,q}^{q,p}\left[(\sqrt{\mu}xt)^\gamma\middle|\begin{matrix}a_1,\ \cdots,\ a_p\\ b_1,\ \cdots,\ b_q\end{matrix}\right]dt$$

$$=x^{-1}\int_0^\infty F(t)(t/x)^{(\gamma-1)/2}G_{p,q}^{q,p}\left[(\sqrt{\mu}t/x)^\gamma\middle|\begin{matrix}a_1,\ \cdots,\ a_p\\ b_1,\ \cdots,\ b_q\end{matrix}\right]dt$$

for all $x>0$.

PROOF. Let $f_0(x)\in L_2(0, \infty)$ be the G-transform of $F(x)$. By Theorem 1 we have

$$(3.14)\qquad \int_0^\infty f_0(t)\, h(xt)\, dt = x^{-1}\int_0^\infty F(t)\, h(t/x)\, dt \text{ for all } x>0,$$

where

$$h(x)=x^{(r-1)/2}\, G^{q,p}_{p,q}\left[(\sqrt{\mu}x)^r \middle| \begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right].$$

But, by hypothesis,

$$(3.15)\qquad \int_0^\infty f(t)\, h(xt)\, dt = x^{-1}\int_0^\infty F(t)\, h(t/x)\, dt \text{ for all } x>0$$

also. Hence

$$(3.16)\qquad \int_0^\infty [f(t)-f_0(t)]\, h(xt)\, dt = 0.$$

By Theorem 3, $f(x)=f_0(x)$ almost everywhere, i.e. $f(x)$ and $F(x)$ are a pair of G-transforms.

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