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ON A GENERALIZED INTEGRAL TRANSFORM

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1. Introduction.

Recently Bora, Kalla and Saxena [2, p.181.3, p.289] have defined an integral transform by means of the integral equation

$$\phi(p,\ q)=\int_0^\infty\int_0^\infty e^{-\frac{px}{2}-\frac{qy}{2}}$$

$$\times H\left[\begin{matrix}\begin{bmatrix} m_1,\ 0 \\ p_1-m_1,\ q_1\end{bmatrix} \\ \begin{pmatrix} m_2,\ n_2 \\ p_2-m_2,\ q_2-n_2\end{pmatrix} \\ \begin{pmatrix} m_3,\ n_3 \\ p_3-m_3,\ q_3-n_3\end{pmatrix}\end{matrix}\left|\begin{matrix}(a_{p_1},\ A_{p_1}),\ (b_{q_1},\ B_{q_1}) \\ (c_{p_2},\ C_{p_2})\ ;(d_{q_2},\ D_{q_2}) \\ (e_{p_3},\ E_{p_3})\ ;(f_{q_3},\ F_{q_3})\end{matrix}\right|\begin{matrix} p^2x^2 \\ \\ q^2y^2\end{matrix}\right] f(x,\ y)dx\,dy$$

$$=H\{f(x,\ y)\ ;p,\ q\}$$

where $H\begin{bmatrix} x \\ y\end{bmatrix}$ denotes the H function in two arguments [4]. The symbol $(a_p,\ A_p)$ stands for p ordered pairs $(a_1,\ A_1),\ \cdots,\ (a_p,\ A_p)$. The transform (1.1) make sense when $p_1\geq m_1\geq 0,\ p_2\geq m_2\geq 0,\ p_3\geq m_3\geq 0,\ q_1\geq 0,\ q_2\geq n_2\geq 0,\ q_3\geq n_3\geq 0,\ q_1+q_2\geq p_1+p_2,$ $q_1+q_3\geq p_1+p_3$, all the p'^s, q'^s, m'^s and n'^s are non negative integers, $|\arg p|<\frac{1}{2}\lambda\pi$, $|\arg q|<\frac{1}{2}\mu\pi$

$$\lambda=\sum_{j=1}^{m_1} A_j-\sum_{j=m_1+1}^{p_1} A_j-\sum_{j=1}^{q_1} B_j+\sum_{j=1}^{m_2} C_j-\sum_{j=m_2+1}^{p_2} C_j-\sum_{j=1}^{n_2} D_j-\sum_{j=n_2+1}^{q_2} D_j>0$$

and

$$\mu=\sum_{j=1}^{m_1} A_j-\sum_{j=m_1+1}^{p_1} A_j-\sum_{j=1}^{q_1} B_j+\sum_{j=1}^{m_3} E_j-\sum_{j=m_3+1}^{p_3} E_j+\sum_{j=1}^{n_3} F_j-\sum_{j=n_3+1}^{q_3} F_j>0$$

$$x^{2\frac{d_i}{D_i}}\ y^{2\frac{f_j}{F_j}}f(x,\ y)\in L(0,\ \delta),\ \delta>0\ (i=1,\ \cdots,\ n_2\ ;j=1,\ \cdots,\ n_3)$$

The object of the present paper is to study the above transform and to establish certain theorems. The first set of theorems which depict certain properties of (1.1) are proved in § 2. In § 3 we establishes its relationship with Varma transform [9, p. 209]

$$g(p)=p\int_0^\infty e^{-\frac{1}{2}pt}(pt)^{m-\frac{1}{2}}\, W_{k,m}(pt)\, h(t)dt, \tag{1.2}$$

while in the last section we prove a relationship between the double Mellin transform of $f(x, y)$ defined by [5].

$$\bar{f}(s,t)=\int_0^\infty\int_0^\infty x^{s-1}\, y^{t-1} f(x, y)dxdy, \tag{1.3}$$

and the double Mellin transform of its image $\phi\{f: p, q\}$ in the generalized transform (1.1), that is

$$\bar{\phi}(s, t)=\int_0^\infty\int_0^\infty p^{s-1}\, q^{t-1}\, \phi[f: p, q]\, dpdq, \tag{1.4}$$

The theorems proved here are quite general and include as particular cases most of the results provided recently by Srivastava [8].

2. Theorems.

In this section we prove the following three theorems that depict some properties of the generalized integral transforms (1.1).

THEOREM 1. *If*

$$H\{f: p, q\}=\int_0^\infty\int_0^\infty e^{-\frac{1}{2}px-\frac{1}{2}qy}\, H\begin{bmatrix} p^2\, x^2 \\ q^2\, y^2 \end{bmatrix} f(x, y)dx\, dy \tag{2.1}$$

then

$$\int_0^\infty\int_0^\infty \frac{H\{f: p, q\}}{p^\alpha\, q^\beta}\, dp\, dq$$

$$=2^{2-\alpha-\beta}\, H\left[\begin{array}{l|l|l} \begin{bmatrix} m_1, 0 \\ p_1-m_1, q_1 \end{bmatrix} & (a_{p_1}, A_{p_1}) : (b_{q_1}, B_{q_1}) & 4 \\ \begin{pmatrix} m_2+1, n_2 \\ p_2-m_2, q_2-n_2 \end{pmatrix} & (\alpha, 2), (c_{p_2}, C_{p_2}) : (d_{q_2}, D_{q_2}) & \\ \begin{pmatrix} m_3+1, n_3 \\ p_3-m_3, q_3-n_3 \end{pmatrix} & (\beta, 2), (e_{p_3}, E_{p_3}) : (f_{q_3}, F_{q_3}) & 4 \end{array}\right]$$

$$\times\int_0^\infty\int_0^\infty \frac{f(x, y)\, dx\, dy}{x^{1-\alpha}y^{1-\beta}}, \tag{2.2}$$

provided that the integral involved are absolutely convergent, $\mathrm{Re}\left(1-\alpha+\frac{d_i}{D_i}\right)>0$ *and* $\mathrm{R}\left(1-\beta+\frac{f_j}{F_j}\right)>0$, $(i=1, \cdots, n_2\,; j=1, \cdots, n_3)$.

PROOF. By the definition of the generalized integral transform (1.1), we have

$$\int_0^\infty\int_0^\infty \frac{H\{f: p, q\}}{p^\alpha q^\beta} dp\, dq$$

$$=\int_0^\infty\int_0^\infty \frac{1}{p^\alpha q^\beta}\left\{\int_0^\infty\int_0^\infty e^{-\frac{1}{2}px-\frac{1}{2}qy}\; H\begin{bmatrix} p^2 x^2 \\ q^2 y^2 \end{bmatrix} f(x, y)\, dx\, dy\right\} dp\, dq$$

Expressing the generalized function of two variables as a Mellin-Barnes type integral, and then changing the order of integration, which is permissible by the absolute convergence of the integrals involved [3, 504], we obtain the desired result on interpreting the definition of $H\begin{bmatrix} x \\ y \end{bmatrix}$

Similar are the proof of the following theorems.

THEOREM 2. *If*

$$H\{f(x, y): p, q\}=\int_0^\infty\int_0^\infty e^{-\frac{1}{2}px-\frac{1}{2}qy}\; H\begin{bmatrix} p^2 x^2 \\ q^2 x^2 \end{bmatrix} f(x, y)\, dx\, dy$$

then

$$H\{f(ax, by)\,; p, q\}=\frac{1}{ab}\; H\left\{ f(x, y): \frac{p}{a}, \frac{q}{b}\right\}$$

provided that the generalized transform of $|f(x, y)|$ *exist and* $a, b>0$.

THEOREM 3. *If*

$$H\{f_j(x, y): p, q\}=\int_0^\infty\int_0^\infty e^{-\frac{1}{2}px-\frac{1}{2}qy}\; H\begin{bmatrix} p^2 x^2 \\ q^2 y^2 \end{bmatrix} f_j(x, y)\, dx\, dy$$

$$(j=1, \cdots, \nu)$$

then for arbitrary multipliers $\Delta_1, \cdots, \Delta_\nu$,

$$\sum_{j=1}^{\nu} \Delta_j H\{f_j(x,y)\,; p, q\}=\sum_{j=1}^{\nu} H\{\Delta_j f_j(x, y): p, q\}$$

provided that the generalized transforms of $|f_j(x, y)|$, $1\leq j\leq\nu$, *exist.*

If we set all the A'^s, B'^s, C'^s, D'^s, E'^s and F'^s equal to one in $H\begin{bmatrix} x \\ y \end{bmatrix}$ then it reduced to S-function of Sharma [7, p.26] and modified Meijer G-function of Agarwal [1, p.536] and consequently the generalized integral transform (1.1) reduces to the one considered recently by Srivastava [8]. Thus the theorems 2, 3 and 4 of him follow as special cases of the theorems 1, 2 and 3 given above.

3. Relationship with the Varma transform.

The following theorem establish a relationship between the generalized integral transform (1.1) and the Varma transform (1.2).

THEOREM 4. *If*

$$H\{f(x,y):p,q\}=\int_0^\infty\int_0^\infty e^{-\frac{1}{2}px-\frac{1}{2}qy}\, H\begin{bmatrix} p^2, x^2 \\ q^2, y^2 \end{bmatrix} f(x,y)\,dx\,dy \tag{3.1}$$

then

$$V_{k,m}[g(t)H\{f(x,y):pt,qt\}:s]$$
$$=\frac{1}{p\,q}\int_0^\infty\int_0^\infty f\left(\frac{x}{p},\frac{y}{q}\right)V_{k,m}\left[g(t)\,e^{-\frac{1}{2}(x+y)t}\, H\begin{bmatrix} x^2\ t^2 \\ y^2\ t^2 \end{bmatrix};s\right]dx$$

provided that the generalized transform of $|f(x,y)|$ *exist, the double integral in* (3.1) *is convergent*

$$\mathrm{Re}\left\{m\pm m+\xi+\frac{d_i}{D_i}+\frac{f_j}{F_j}+1\right\}>0$$
$$(i=1,\cdots,n_2; j=1,\cdots,n_3)$$

Re $(p-2\delta)>0$, $p,q>0$, *where* $g(t)=0\ (t^{\xi})$ *for small* $t>0$, *and* $g(t)=0\ (t^{\eta}e^{st})$ *for large* t

PROOF. By (1.1) and (1.2) we have

$$V_{k,m}[g(t)H\{f(x,y):pt,qt\}:s]$$
$$=s\int_0^\infty e^{-\frac{1}{2}\delta t}\,(\delta t)^{m-\frac{1}{2}}W_{k,m}(\delta t)\ g(t)\left(\int_0^\infty\int_0^\infty e^{-\frac{1}{2}(px+qy)t}\, H\begin{bmatrix} p^2t^2x^2 \\ q^2t^2y^2 \end{bmatrix} f(x,y)\,dx\,dy\right)dt$$
$$=\frac{1}{pq}\int_0^\infty\int_0^\infty f\left(\frac{x}{p},\frac{y}{q}\right)\left(s\int_0^\infty e^{-\frac{1}{2}\delta t}\,(\delta t)^{m-\frac{1}{2}}\,W_{k,m}(\delta t)\,g(t)\,e^{-\frac{1}{2}(x+y)t}\,H\begin{bmatrix} x^2\ t^2 \\ y^2\ t^2 \end{bmatrix}dt\right)dx\,dy.$$

The change of order of integration is justified by de la Vallée Poussins theorem [3, p.504] under the conditions stated with the theorem.

4. Relationship with the Double Mellin Transform.

By (1.1) and (1.4) we have

$$\bar{\phi}(s,t)=\int_0^\infty\int_0^\infty p^{s-1}q^{t-1}H\{f(x,y):p,q\}\,dp\,dq$$

$$=\int_0^\infty\int_0^\infty p^{s-1}q^{t-1}\left[\int_0^\infty\int_0^\infty e^{-\frac{px}{2}-\frac{qy}{2}}H\begin{bmatrix}p^2x^2\\ 2\ \ 2\\ q^2y^2\end{bmatrix}f(x,y)\,dx\,dy\right]dp\,dq$$

$$=\int_0^\infty\int_0^\infty x^{-s}y^{-t}f(x,y)\,dx\,dy\int_0^\infty\int_0^\infty u^{s-1}v^{t-1}e^{-\frac{1}{2}(u+v)}H\begin{bmatrix}u^2\\ 2\\ v^2\end{bmatrix}du\,dv$$

on changing the order of integration and a little simplification. Since the x, y integral and u, v integral above are independent of each other, we are lead to the following theorem.

THEOREM 5. *If the double Mellin transform of* $|f(x,y)|$ *and* $|H\{f(x,y):p,q\}|$ *exist, then*

$$\bar{\phi}(s,t)=\bar{f}(1-s,1-t)\phi(s,t)$$

where

$$\phi(s,t)=2^{s+t}\,H\left[\begin{matrix}\begin{bmatrix}m_1,\ 0\\ p_1-m_1,\ q_1\end{bmatrix}\\ \begin{pmatrix}m_2+1,\ n_2\\ p_2-m_2,\ q_2-n_2\end{pmatrix}\\ \begin{pmatrix}m_3+1,\ n_3\\ p_3-m_3,\ q_3-n_3\end{pmatrix}\end{matrix}\left|\begin{matrix}(a_{p_1},A_{p_1}):(b_{q_1},B_{q_1})\\ (1-s,2),(c_{p_2},C_{p_2}):(d_{p_1},D_{q_2})\\ (1-t,2),(e_{p_3},E_{p_3}):(f_{q_3},F_{q_3})\end{matrix}\right|\begin{matrix}4\\ \\ 4\end{matrix}\right]$$

and provided that $|H\{f(x,y):p,q\}|$ *exist,*

$$R\left(s+2\frac{d_i}{D_i}\right)>0\ (i=1,\cdots,n_2),\quad R\left(t+2\frac{f_j}{F_j}\right)>0\ (j=1,\cdots,n_3)$$

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