Kyungpook Math. J.
Volume 13, Number 2
December, 1973

SHEAVES ON PRIME SPECTRUMS OF QUASI *B*-RINGS

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1. Introduction.

Let A be a commutative ring with 1, the prime spectrum Spec(A) of A is the topological space with Zariski topology. That is, Spec(A) is the set of all prime ideals of A whose closed sets are of the form $V(E)$, where E is a subset of A and $V(E)$ the set of all prime ideals of A which contain E ([1], [7]).

The object of this paper is to define a quasi B-ring (§2), since we can define a sheaf on the prime spectrum of such a ring by using basic open sets (§2 and §4).

We also give an example of a quasi B-ring in §3, and finally prove that

$$H^0_\varphi(X, \bar{S}) \cong \Gamma_\varphi(X: \prod_{x\in X} A_{p_x})$$

$$H^n_\varphi(X, \bar{S}) = 0 \ (n > I).$$

where A is a quasi B-ring, X=Spec(A), $\bar{S}$=the sheaf on X is defined by using basic open sets of X, $\{H^n\}$=the cohomologys functors and φ a family of supports on X (Theorem 3 in §4).

Throughout this paper the word "ring" shall mean a commutative ring with 1.

2. Properties of Spec(A).

Let A be a ring, For each $f \in A$ we denote the complement of $V(f)$ in X=Spec(A) by X_f. Then every open set in X is the union of open sets X_f, since

$$X - V(E) = X - \bigcap_{f\in E} V(f) = (\bigcap_{f\in E} V(f))^c = \bigcup_{f\in E} V(f)^c = \bigcup_{f\in E} X_f,$$

where $V(f)^c$ is the complement of $V(f)$ and E is a subset of A. This means that the family of open sets X_f is a basic of open sets for the Zariski topology ([6]). The sets X_f are called *basic open sets* of X.

PROPOSITION 1. $X=\mathrm{Spec}(A)$ *is compact. More generally, each* X_f $(f\in A)$ *is compact.*

PROOF. Assume $\mathscr{U}=\{U_i\}_{i\in I}$ is open covering of X. Since $\{X_f\}_{f\in A}$ is a base, for each $x\in U_i$ there exists an open set X_{f_i} with $x\in X_{f_i}\subset U_i$. Therefore we have an open covering $\{X_{f_i}\}_{i\in J}$ of X which is induced by $\mathscr{U}$. Thus, it follows from $X=\bigcup_{i\in J}X_{f_i}\in\bigcup_{i\in J}V(f_i)^c=(\bigcap_{i\in J}V(f_i))^c=(V(\bigcup_{i\in J}f_i))^c$ that $V(\bigcup_{i\in J}f_i)=\phi=V(1)$, and therefore there exist $g_1, \cdots, g_n$ in A such that $\sum_{i=1}^{n}g_if_i=1$. where $f_1, \cdots, f_n\in\bigcup_{i\in J}f_i$. This implies that $V(\{f_1, \cdots, f_n\})=V(1)$, and thus $X=X_{f_1}\cup\cdots\cup X_{f_n}$. The second part of this proposition is proved by the same way as above.

We give the following conditions into $\mathrm{Spec}(A)=X$:

(C_1) Each element of X is closed.

(C_2) Each X_f is open and closed in X.

(C_3) Each open set of X is a compact subset.

THEOREM 1. *Let* $\mathrm{Spec}(A)$ *satisfy the above conditions* (C_1), (C_2), (C_3). *Then* $\mathrm{Spec}(A)$ *is a discrete topological space.*

PROOF. Let x be a point of X. By (C_1) x is a closed set of X. We have to prove that x is also open in X. Since X-x is open there exist $f_1, f_2, \cdots, f_n$ in A such that $X-x=X_{f_1}\cup\cdots\cup X_{f_n}$, Since each X_{f_i} is closed in X (by (C_2)), $X_{f_1}\cup\cdots\cup X_{f_n}=X-x$ is closed. Therefore x is open.

Note that $x=p_x$(: a prime ideal of A) is closed in X if and only if p_x is a maximal ideal of A. We give one more condition (C_4) below into $\mathrm{Spec}(A)$:

(C_4) a finite union $X_{f_1}\cup\cdots\cup X_{f_n}$($f_i\in A$ for $i=1, \cdots, n$) is equal to X_f for some $f\in A$.

A ring satisfying conditions (C_1)-(C_4) above is called a *quasi B-ring* (for examples, see the next section).

3. Boolean rings.

Let A be a ring. If for each $x\in A$ $2x=0$ and $x(1+x)=0$ then A is called a *Boolean ring*. In a Boolean ring A, for each $x\in A$ $x^2=x$, because of $2x=0$ and $x(1+x)=0$, $x^2=-x$. Therefore we have that A is a Boolean ring and for each $x\in A$ $x=x^2$.

PROPOSITION 2. *Let A be a Boolean ring. Then*
i) *every prime ideal p of A is maximal and A/p is a field with two elements,*
ii) *every finitely generated ideal in A is principal ideal.*

PROOF. Suppose a prime ideal p of A. Then there exists an element x in A such that $x \notin p$. By the definition, $x(1+x)=0 \in p$ which implies that $1+x \in p$. Thus $1 \equiv x \bmod p$ and therefore $A/p=\{[0], [1]\}$, where $[0]$ and $[1]$ are the classes containing 0 and 1, respctively. We complete the proof of i). For proof of ii), it suffices to consider an ideal Ax_1+Ax_2, where $x_1, x_2 \in A$. Put $y=x_1+x_2+x_1x_2$, then we get $x_1=x_1y$, $x_2=x_2y$ and $x_1x_2=x_1x_2y$ and therefore $Ax_1+Ax_2=Ay$.

For a Boolean ring A, we shall prove the following.

PROPOSITION 3. *Let $X=$ Spec(A). Then*

i) *Each $x \in X$ is closed in X.*
ii) *For each $f \in A$, the set X_f is open and closed in X.*
iii) *For $f_1, \cdots, f_n \in A$ there exists an element f in A such that $X_{f_1} \cup \cdots \cup X_{f_n}=X_f$*
iv) *X is a compact Hausdorff space.*

PROOF. i) is easily proved by i) of the preceding proposition.
ii): There exists a maximal ideal p of A such that $f \in p$. In this case $g=1+f$ is not in p. By the definition of a Boolean ring we have $f+g=1$. Thus

$$V(\{f, g\})=V(1)=V(f) \cap V(g)=\phi,$$
$$(V(f) \cap V(g))^c=V(f)^c \cup V(g)^c=\phi^c=X, \ X_f \cup X_g=X$$

Since $X_f \cap X_g=X_{fg}$ and $f \circ g \equiv f(1+f)=0$ we have $X-X_g=X_f-X_{fg} \equiv X_f-X_0=X_f$ and therefore X_f is closed (Note: $X_0=X-V(0)=\phi$).
iii): $X_{f_1} \cup \cdots \cup X_{f_n}=V(f_1)^c \cup \cdots \cup V(f_n)^c=(V(f_1) \cap \cdots \cap V(f_n))^c=V(\{f_1, \cdots, f_n\})^c$ $=V(A_{f_1}+\cdots+A_{f_n})^c$. By ii) of Proposition 2, there exists an element f in A such that $Af=Af_1+\cdots+Af_n$, and thus $X_{f_1} \cup \cdots \cup X_{f_n}=V(Af)^c=V(f)^c=X_f$.
iv): Note that each element x of X is the maximal ideal p_x of A. Take two elements x and y in X with $x \neq y$. Then $p_x \neq p_y$. We can choose two elements f and g in A such that $f \in p_x$, $g \in p_y$, $f \notin p_y$, $g \notin p_x$ and $f+g=1$ (see proof of ii of this proposition). In this case, X_f is an open neighborhood of y, X_g is an open neighborhood of x, and $X_f \cap X_g=X_{fg}=\phi$. Therefore X is a Hausdorff space.

Let A be a finite Boolean ring. By the preceding proposition, A is a quasi B-ring.

4. The sheaf of a quasi B-ring.

In this section we only deal with quasi B-rings. Therefore the word "ring" means a quasi B-ring. Let A be a ring, and let $X=\mathrm{Spec}(A)$. By the definition in the section 2, each open set in $\mathrm{Spec}(A)$ is of the form X_f for some $f\in A$. We want to define a presheaf on $\mathrm{Spec}(A)$, using basic open sets.

PROPOSITION 4. *If* $X_g\subset X_f$ *in* $\mathrm{Spec}(A)$, *then there exists an equation of the form* $g^n=uf$ *for some integer* $n>0$ *and some* $u\in A$

PROOF. $X_g\subset X_f$ implies that $X-V(g)\subset X-V(f)$. From $V(f)\subset V(g)$ we know that there exist $u\in A$ and $n>0$ such that $g^n=uf$.

We put $S=\{f, f^2, \cdots\}$ and $S^{-1}A=A_f$. The process of passing from A to A_f is *localization* at $\{f, f^2, \cdots\}$.

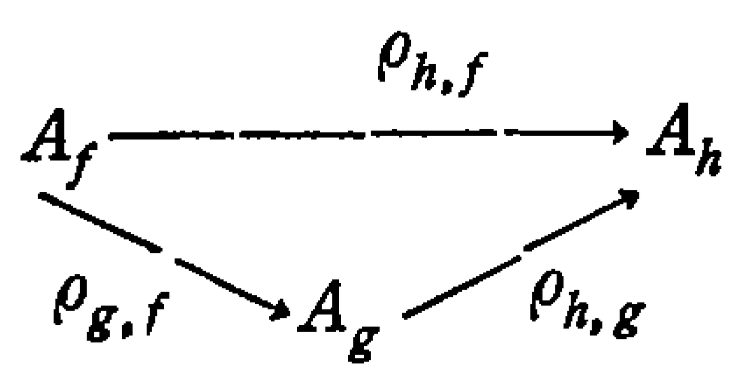
Using the preceding proposition we define a A-module map $\rho_{g,f}: A_f\to A_g$ by $\rho_{f,g}(a/f^m)=u^m a/g^{nm}$ where $X_g\subset X_f$ and $a\in A$. In this case, if $X_g=X_f$ then $A_g=A_f$, because there exist u_1, u_2 in A such that $g^m=u_1f$ and $f^n=u_2g$ for some m, $n>0$, and

$$[\rho_{f,g}\circ\rho_{g,f}](a/f^p)=\frac{u_1^p u_2^{mp} a}{f^{mnp}}=\frac{a}{f^p}.$$

We define two categories X_A and M_A as follows:
X_A=the category of all basic open sets of $X=\mathrm{Spec}\ (A)$ and inclusion map,
M_A=the category of all A-modules and A-module maps.

PROPOSITION 5. *The contravariant functor* $S:X_A\to M_A$ *which is defined by* $S(X_f)=A_f$ *and* $S(i)=\rho_{g,f}$ *for any inclusion map* $i:X_g\to X_f$ *is a presheaf on* X. *Furthermore, for every point* $x\in X$, *the stalk* S_x *at* x *is isomorphic to* A_{p_x} *where* $x=p$ *is a prime ideal of* A.

PROOF. For the first part of this proposition, it suffices to prove that the diagram



is commutative, where $X_h \subset X_g \subset X_f$, because $\rho_{f,f}$ is the identity map ([5], [8]). From the relation $X_h \subset X_g \subset X_f$ we have equations $g^n = u_1 f$, $h^p = u_2 g$ and $h^q = u_3 f$, where $u_1, u_2, u_3 \in A$ and $n, p, q > 0$. For any element $\frac{a}{f^m} \in A_f$,

$$\rho_{h,g} \circ \rho_{g,f}\left(\frac{a}{f^m}\right) = \frac{u_2^{nm} u_1^m a}{h^{pnm}}, \rho_{h,f}\left(\frac{a}{f^m}\right) = \frac{u_3^m a}{h^{qm}}.$$

Since we have

$$a u_2^{nm} u_1^m h^{qm} = a u_3^m u_2^{nm} u_1^m f^m = a u_3^m u_2^{mn} g^{nm} = a u_3^m h^{pnm}$$

we get

$$\frac{u_2^{nm} u_1^m a}{h^{pnm}} = \frac{u_3^m a}{h^{qm}} \text{ in } A_h.$$

Thus the above diagram is commutative.

Next, we have to prove that

$$\varinjlim_{x \in X_f} A_f \cong A_p.$$

Noting that the following are equivalent

(i) $x \in X_f$, (ii) $x \in X - V(f)$, (iii) $f \in A - p_x$,

we know that there is the canonical map $A_f \to A_p$ assigns $\frac{a}{f_n}$ with $\frac{a}{f^n}$. Thus we have to prove that

i) If $\frac{a}{f} = \frac{b}{g}$ in A_p where $\frac{a}{f} \in A_f$ and $\frac{b}{g} \in A_g$, then there exists α in A such that $x \in X_\alpha \subset X_f$, $X_\alpha \subset X_g$ and $\rho_{\alpha,f}\left(\frac{a}{f}\right) = \rho_{\alpha,g}\left(\frac{b}{g}\right)$ in X_α,

ii) Conversely, if $\frac{a}{f} \in A_f$ and $\frac{b}{g} \in A_g$ are equal in some A_h such that $x \in X_h \subset X_f$ and $X_\alpha \subset X_g$ then $\frac{a}{f} = \frac{b}{g}$ in A_p.

Proof of i): Note that

$$\frac{a}{f} = \frac{b}{g} \text{ in } A_p \text{ iff there exists } h \in A - p_x \text{ such that } h(ag - bf) = 0$$

There are two maps $\rho_{fgh,f}: A_f \to A_{fgh}$ and $\rho_{fgh,g}: A_g \to A_{fgh}$ sucht hat $\rho_{fgh,f}\left(\frac{a}{f}\right) = \frac{agh}{fgh}$ and $\rho_{fgh,g}\left(\frac{b}{g}\right) = \frac{bfh}{fgh}$, respectively. Since $agh = bfh$ we have $\frac{agh}{fgh} = \frac{bfh}{fgh}$ in A_{fgh}. Put $\alpha = fgh$ then i) has been proved.

Proof of ii). From $\rho_{h,f}\left(\frac{a}{f}\right)=\rho_{h,g}\left(\frac{b}{g}\right)$ we get $\rho_{fgh,f}\left(\frac{a}{f}\right)=\frac{agh}{fgh}=\rho_{fgh,g}\left(\frac{b}{g}\right)$ $=\frac{bfh}{fgh}$ in A_{fgh}. Thus we have the equation $(fgh)^m(agh-bfh)=0$ i. e. $(fgh)^m$ $h(ag-bf)=0$. Since f, g and h are in $A-p_x(fgh)^m h$ is also in $A-p_x$ and therefore $\frac{a}{f}=\frac{b}{g}$ in A_{p_x}.

THEOREM 2. *The presheaf S as above is the sheaf* $(\bar{S}\ \pi,\ X)$ *on* Spec$(A)=X$ *such that for each* $x\in X$, $\pi^{-1}(x)=A_{p_x}$ *and for each open set U of X*, $S(U)=\bar{S}(U)$, *where* $x=p_x$, π *the canonical projection and* $\bar{S}(U)=$*the set of all sections of* $\bar{S}$ *over U*.

PROOF. We want to prove that for $X=\bigcup_{i\in I}X_{f_i}$ and $\left\{\frac{a_i}{f_i}\in A_{f_i}\right\}_{i\in I}$ with $\rho_{f_if_j,f_i}\left(\frac{a_i}{f_i}\right)$ $=\rho_{f_if_j,f_j}\left(\frac{a_j}{f_j}\right)$, there exists a unique element a in A such that $\rho_{f_i,1}(a)=\frac{a_i}{f_i}$ for all $i\in I$ (note: $X=X-V(1)=X_1\supset X_{f_i}$). In order to do this, it suffices to deal with the cases that f_i has no a_i as a factor for all $i\in I$. Assume f_i is not divisor of a_i, then

$$\frac{a_if_j}{f_if_j}=\frac{a_jf_i}{f_if_j} \qquad \text{in } A_{f_if_j}\ (\text{note}: X_{f_i}\cap X_{f_j}=X_{f_if_j}).$$

Therefore there is a positive integer $n\geq 1$ such that $(f_if_j)^n(a_if_j-a_jf_i)=0$ and we have $(f_if_j)^n a_if_j=(f_if_j)^n a_jf_i=0$, because of $a_if_j\neq a_jf_i$ except $a_if_j=0=a_jf_i$. This implies that

$$\frac{a_if_j}{f_if_j}=\frac{a_jf_i}{f_if_j}=0 \text{ in } A_{f_if_j}.$$

Therefore

$$\frac{a_if_j}{f_if_j}=\frac{a_jf_i}{f_if_j}\neq 0$$

in $A_{f_if_j}$ implies that there exists a unique element a in A such that $a_i=af_i$ for all $i\in I$. For all $i\in I$. $\rho_{f_i,1}(a)=\frac{af_i}{f_i}$, where $\rho_{f_i,1}: A\rightarrow A_{f_i}$ and therefore we complete our proof (because the presheaf S satisfies the conditions (S_1) and (S_2) in pages 5-6 of [5]).

Recall that Spec(A) is a discrete topological space (§2). This implies that $\bar{S}$ is also a discrete topological space, and therefore we have the sheaf isomorphism $\bar{S}\approx\prod_{i\in X}A_{p,}$(with the discrete topology) (for details see [5] or [8]). In consequence the sheaf $\bar{S}$ on Spec(A) is *flably*. Let φ be a family of supports on X, and Γ_{φ} (U, F) be the set of all sections of a sheaf F on U over X, where U is an open set of X. Since each $\bar{F}$ has an injective resolution we can define the cohomology functors of $\bar{F}$ as follows:

$$H^n_{\varphi}(X, \bar{F})=R^n\Gamma\varphi(X, \bar{F}) \quad (n\geq 0)$$

where $\{R_n\}_{n\geq 0}$ are the right derived functors of the contravariant functors Γ_{φ}(for details see [5] or [8]), Using that every flabby sheaf is φ-acyclic for any φ the following is clear (see p. 35 of [5]).

THEOREM 3. *Let A be a quasi B-ring. There exists a unique sheaf* $\bar{S}$ *on* $X=$ Spec(A) *up to isomorphism such that for each open set* $U(=X_f$ *for some* $f\in A)$ $\bar{S}(U)\cong A_f$, *which is flabby. That is, for any family* φ *of supports on* X

$$H^0_{\varphi}(X, \bar{S})\cong\Gamma_{\varphi}(X: \prod_{x\in X}A_{p,}), \quad H^n_{\varphi}(X, \bar{S})=0, \ n>1$$

Moreover, if φ *is the set of all closed subsets of* X *then* $H^0_{\varphi}(X: \bar{S})=H^0(X,\bar{S})$ $\cong\prod_{x\in X}A_{p.}$

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