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A NOTE ON MILDLY NORMAL SPACES

By Takashi Noiri

1. Introduction.

In 1973, M.K. Singal and Asha Rani Singal [5] introduced the concept of mildly normal spaces as a generalization of normal spaces and obtained several properties of such a space. This space is equivalent to the space which in 1971 C. Wenjen [7] has called a *pseudo-normal space*. In [5], among others, the following theorem is stated without the proof.

THEOREM A. *Every closed, continuous and open image of a mildly normal space is mildly normal.*

The purpose of the present note is to improve the above theorem. The concept of almost-continuity, due to M.K. Singal and Asha Rani Singal [6], is useful for the purpose.

Let A be a subset of a topological space. We denote the closure of A and the interior of A by Cl A and Int A respectively. A is said to be *regularly open* if Int Cl $A=A$, and *regularly closed* if Cl Int $A=A$. RO(X) (RC(X)) will denote the family of all regularly open (regularly closed) sets in a topological space X. By spaces we mean topological spaces, by $f: X \to Y$ we denote a mapping (not necessarily continuous) f of a space X into a space Y.

2. Preliminaries.

DEFINITION 1. A mapping $f: X \to Y$ is said to be *almost-continuous*(*θ-continuous*) if for each point $x \in X$ and each neighborhood V of $f(x)$ in Y, there exists a neighborhood U of x such that $f(U) \subset \text{Int Cl } V$ ($f(\text{Cl } U) \subset \text{Cl } V$) [6]([2]).

REMARK 1. We have *continuity $\Rightarrow$ almost-continuity $\Rightarrow$ θ-continuity*, but none of these implications is reversible [3], [4], [6].

REMARK 2. The almost-continuity of a mapping $f: X \to Y$ is characterized by the following statements: 1) For each $V \in \text{RO}(Y)$, $f^{-1}(V)$ is open in X; 2) For each $B \in \text{RC}(Y)$, $f^{-1}(B)$ is closed in X [6].

DEFINITION 2. A mapping $f: X \to Y$ is said to be *almost-open* (*almost-closed*) if for each $U \in \mathrm{RO}(X)$ ($\mathrm{RC}(X)$), $f(U)$ is open (closed) in Y [6].

REMARK 3. Every open (closed) mapping is almost-open (almost-closed), but the converse is not necessarily true [6].

LEMMA 1. *If $f: X \to Y$ is almost-open and θ-continuous, then f is almost-continuous.*

PROOF. For each point $x \in X$ and each neighborhood V of $f(x)$ in Y, there exists an open neighborhood U of x such that $f(\mathrm{Cl}\ U) \subset \mathrm{Cl}\ V$ because f is θ-continuous. Since f is almost-open and $\mathrm{Int\ Cl}\ U \in \mathrm{RO}(X)$, $f(\mathrm{Int\ Cl}\ U)$ is open and hence we have $f(U) \subset f(\mathrm{Int\ Cl}\ U) \subset \mathrm{Int}[f(\mathrm{Cl}\ U)]$. Thus we obtain $f(U) \subset \mathrm{Int\ Cl}\ V$. This shows that f is almost-continuous.

LEMMA 2. *If a mapping $f: X \to Y$ is almost-continuous and almost-open, then*
1) *For each* $V \in \mathrm{RO}(Y)$, $f^{-1}(V) \in \mathrm{RO}(X)$:
2) *For each* $B \in \mathrm{RC}(Y)$, $f^{-1}(B) \in \mathrm{RC}(X)$.

PROOF. 1) If $V \in \mathrm{RO}(Y)$, then $f^{-1}(V)$ is open and hence we have $f^{-1}(V) \subset \mathrm{Int\ Cl}\ f^{-1}(V)$. On the other hand, since f is almost-continuous and $\mathrm{Cl}\ V \in \mathrm{RC}(Y)$, $f^{-1}(\mathrm{Cl}\ V)$ is closed and hence we have $\mathrm{Int\ Cl}\ f^{-1}(V) \subset \mathrm{Cl}\ f^{-1}(V) \subset f^{-1}(\mathrm{Cl}\ V)$. Moreover, since f is almost-open and $\mathrm{Int\ Cl}\ f^{-1}(V) \in \mathrm{RO}(X)$, $f[\mathrm{Int\ Cl}\ f^{-1}(V)]$ is open. Hence we have $f[\mathrm{Int\ Cl}\ f^{-1}(V)] \subset \mathrm{Int\ Cl}\ V = V$. Thus we obtain $\mathrm{Int\ Cl} f^{-1}(V) \subset f^{-1}(V)$. This completes the proof of 1).

2) The proof of 2) follows easily from 1) and the following two facts: (a) $f^{-1}(Y-V) = X - f^{-1}(V)$ for each subset $V \subset Y$; (b) $V \in \mathrm{RO}(Y)$ if and only if $Y - V \in \mathrm{RC}(Y)$.

REMARK 4. Every almost-continuous, almost-open and almost-closed mapping is not necessarily continuous, as the following example shows.

EXAMPLE 1. Let X be the set of real numbers and $\mathscr{T}_X$ the countable complement topology for X. Let Y be a set $\{a, b\}$ of distinct points and $\mathscr{T}_Y = \{Y, \{a\}, \phi\}$. We define a mapping $f: (X, \mathscr{T}_X) \to (Y, \mathscr{T}_Y)$ as follows: $f(x) = a$ if x is rational and $f(x) = b$ if x is irrational. Then f is almost-continuous, but not continuous [6, Example 2.1]. Moreover, since $\mathrm{RO}(X, \mathscr{T}_X) = \mathrm{RC}(X, \mathscr{T}_X) = \{X, \phi\}$, f is almost-open and almost-closed.

The following lemma is a slight modification of [1, p.96, Exercise 10].

LEMMA 3. *A surjective mapping $f: X \to Y$ is almost-closed if and only if for any subset $S \subset Y$ and any $U \in \mathrm{RO}(X)$ containing $f^{-1}(S)$, there exists an open set V in Y such that $S \subset V$ and $f^{-1}(V) \subset U$.*

PROOF. Necessity. Suppose f is almost-closed. For any subset $S \subset Y$ and any $U \in \mathrm{RO}(X)$ containing $f^{-1}(S)$, let us put $V = Y - f(X-U)$. Then, since $f^{-1}(S) \subset U$, we have $S \subset V$. Since f is almost-closed and $U \in \mathrm{RO}(X)$, V is open in Y. By a straightforward calculation we obtain $f^{-1}(V) \subset U$.

Sufficiency. Suppose $A \in \mathrm{RC}(X)$ and $y \in Y - f(A)$. Then we have $f^{-1}(y) \subset X - A \in \mathrm{RO}(X)$. By the hypothesis there exists an open set V in Y such that $y \in V$ and $f^{-1}(V) \subset X - A$. Thus we obtain $y \in V \subset Y - f(A)$. This implies that $Y - f(A)$ is open in Y. Hence $f(A)$ is closed. Consequently, f is almost-closed.

3. Mildly normal spaces.

DEFINITION 3. A space X is said to be *mildly normal* if for every pair of disjoint F_1 and $F_2 \in \mathrm{RC}(X)$, there exist disjoint open sets U_1 and U_2 such that $F_1 \subset U_1$, $F_2 \subset U_2$ [5].

THEOREM B (Singal and Singal, [5]). *For a space X, the following are equivalent:*

(a) X is mildly normal.

(b) For any $A \in \mathrm{RC}(X)$ and any $V \in \mathrm{RO}(X)$ such that $A \subset V$, there exists an open set U such that $A \subset U \subset \mathrm{Cl}\, U \subset V$.

(c) For any $A \in \mathrm{RC}(X)$ and any $V \in \mathrm{RO}(X)$ such that $A \subset V$, there exists $U \in \mathrm{RO}(X)$ such that $A \subset U \subset \mathrm{Cl}\, U \subset V$.

THEOREM 1. *The almost-continuous almost-closed image of a normal space is mildly normal.*

PROOF. Let X be a normal space (not necessarily T_1) and $f: X \to Y$ be an almost-continuous and almost-closed surjection. Suppose B_1 and B_2 are disjoint regularly closed sets in Y. Since f is almost-continuous, $f^{-1}(B_1)$ and $f^{-1}(B_2)$ are disjoint closed sets in X. By the normality of X, there exist disjoint open sets U_1 and U_2 such that $f^{-1}(B_j) \subset U_j$ for $j=1,2$. Since U_1 and U_2 are disjoint open, $\mathrm{Int\,Cl}\, U_1$ and $\mathrm{Int\,Cl}\, U_2$ are disjoint regularly open sets such that $f^{-1}(B_j) \subset U_j \subset \mathrm{Int\,Cl}\, U_j$ for $j=1, 2$. Since f is almost-closed, by Lemma 3, there exists an open set V_j in Y such that $B_j \subset V_j$ and $f^{-1}(V_j) \subset \mathrm{Int\,Cl}\, U_j$ for $j=1, 2$. Since f is surjective, V_1 and V_2 are disjoint. This implies that Y is mildly normal.

THEOREM 2. *The mildly normality is invariant under θ-continuous, almost-open and almost-closed surjections.*

PROOF. Let X be a mildly normal space and $f: X \to Y$ be a θ-continuous, almost-open and almost-closed surjection. Let us suppose $A \in \mathrm{RC}(Y)$, $V \in \mathrm{RO}(Y)$ and $A \subset V$. Since f is θ-continuous and almost-open, by Lemma 1 and 2, we have $f^{-1}(A) \in \mathrm{RC}(X)$, $f^{-1}(V) \in \mathrm{RO}(X)$ and $f^{-1}(A) \subset f^{-1}(V)$. Since X is mildly normal, by Theorem B, there exists $W \in \mathrm{RO}(X)$ such that $f^{-1}(A) \subset W \subset \mathrm{Cl}\, W \subset f^{-1}(V)$. Hence we have $A \subset f(W) \subset f(\mathrm{Cl}\, W) \subset V$ because f is surjective. Since f is almost-open and almost-closed, $f(W)$ is open and $f(\mathrm{Cl}\, W)$ is closed. Therefore, let us put $U = f(W)$, and we have $A \subset U \subset \mathrm{Cl}\, U \subset V$. By Theorem B, we observe that Y is mildly normal.

COROLLARY (Singal and Singal, [5]). *The mildly normality is invariant under continuous, open and closed surjections.*

Miyakonojo Technical College
Miyakonojo, Miyazaki-ken
Japan

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