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PRESERVATION OF CONNECTEDNESS UNDER EXTENSIONS OF TOPOLOGIES

By Donald F. Reynolds[1]

1. Introduction.

Throughout, $(X, \mathscr{T})$ will be a topological space.

DEFINITION. Let $\mathfrak{A}=\{A_\alpha | \alpha \in \Gamma\}$ be a collection of subsets of X. The topology $\mathscr{T}(\mathfrak{A})$ generated by $\mathscr{T} \cup \mathfrak{A}$ is the *extension of the topology* $\mathscr{T}$ *to the collection* $\mathfrak{A}$. If $\mathfrak{A}$ is a singletion (finite, infinite) set, then $\mathscr{T}(\mathfrak{A})$ is called a *simple (finite, infinite) extension* of the topology $\mathscr{T}$. Simple extensions are denoted by $\mathscr{T}(A)$, rather than $\mathscr{T}(\{A\})$.

Both Levine [2] and Borges [1] have raised the following question: Under what conditions will a given topological property P be preserved under an extension of the topology?

Levine [2, Theorem 9] has shown that connectedness is preserved under a simple extension of the topology to a dense, connected set A.

It is easy to see that in order to remove the requirement that A be dense, it is necessary to assume some additional hypotheses. For example, Borges [1, Theorem 4.2] shows that connectedness is preserved if A is not $\mathscr{T}$-closed and both A and $X-A$ are connected. On the other hand, our Corollary 1 shows that Levine's requirement that A be connected may be removed without the introduction of additional hypotheses, i.e., connectedness is preserved under an extension of the topology to any dense set.

2. Preliminary Lemma.

We first introduce a lemma which gives a useful representation of open sets in an extension topology.

1 Portions of this paper appear in the author's doctoral dissertation, written at Texas Christian University originally under the direction of the late Professor Hisahiro Tamano and completed under Professor Arnold R. Vobach of the University of Houston.

LEMMA. *Let $\mathfrak{A}=\{A_\alpha|\alpha\in\Gamma\}$ be a collection of subsets of X and let $\mathscr{P}_f(\Gamma)$ denote the collection of all finite subsets of Γ. Then each $O\in\mathscr{T}(\mathfrak{A})$ can be expressed in the form $O=\bigcup\{O^s\cap(\bigcap_{\alpha\in s}A_\alpha))|s\in\mathscr{P}_f(\Gamma)\}$, where $O^s\in\mathscr{T}$ for all $s\in\mathscr{P}_f(\Gamma)$.*

PROOF. This follows from the fact that $\mathscr{T}\cup\{A_\alpha|\alpha\in\Gamma\}$ is a subbasis for $\mathscr{T}(\mathfrak{A})$.

3. Main Theorem.

THEOREM. *Let $(X,\ \mathscr{T})$ be connected. Let $\mathfrak{A}=\{A_\alpha|\alpha\in\Gamma\}$ be a collection of subsets of X such that finite intersections of members of $\mathfrak{A}$ are dense in $(X,\ \mathscr{T})$. Then $(X,\ \mathscr{T}(\mathfrak{A}))$ is connected.*

PROOF. Suppose not. Then there exist two non-empty $\mathscr{T}(\mathfrak{A})$-open sets, N_1 and N_2, such that $N_1\cap N_2=\phi$ and $N_1\cup N_2=X$. By the connectedness of $(X,\mathscr{T})$, at least one of these is not $\mathscr{T}$-open, say N_1. Then there exists $x\in N_1$ such that each $\mathscr{T}$-open neighborhood of x intersects $X-N_1=N_2$.

Let V be an arbitrary $\mathscr{T}(\mathfrak{A})$-open neighborhood of x. By the above lemma, $V=\bigcup\{V^s\cap(\bigcap_{\alpha\in s}A_\alpha)|s\in\mathscr{P}_f(\Gamma)\}$. Thus for some $s=s(x)\in\mathscr{P}_f(\Gamma)$, we have that $x\in V^{s(x)}\cap(\bigcap_{\alpha\in s(x)}A_\alpha)$, where $V^{s(x)}\in\mathscr{T}$. But $V^{s(x)}\in\mathscr{T}$ implies $V^{s(x)}\cap N_2\neq\phi$. Let $y\in V^{s(x)}\cap N_2$. Since $N_2\in\mathscr{T}(\mathfrak{A})$, again we can apply the lemma to obtain $N_2=\bigcup\{N_2^s\cap(\bigcap_{\alpha\in s}A_\alpha)|s\in\mathscr{P}_f(\Gamma)\}$. Then $y\in N_2$ implies that for some $s=s(y)\in\mathscr{P}_f(\Gamma)$, $y\in N_2^{s(y)}\cap(\bigcap_{\alpha\in s(y)}A_\alpha)$, where $N_2^{s(y)}\in\mathscr{T}$. Thus $y\in V^{s(x)}\cap N_2^{s(y)}$, a $\mathscr{T}$-open set. By hypothesis, $\bigcap\{A_\alpha|\alpha\in s(x)\cup s(y)\}$ is dense in $(X,\ \mathscr{T})$ so that $\phi\neq V^{s(x)}\cap N_2^{s(y)}\cap(\bigcap\{A_\alpha|\alpha\in s(x)\cup s(y)\})=[V^{s(x)}\cap(\bigcap_{\alpha\in s(x)}A_\alpha)]\cap[N_2^{s(y)}\cap(\bigcap_{\alpha\in s(y)}A_\alpha)]\subset V\cap N_2$. Thus x lies in the $\mathscr{T}(\mathfrak{A})$-closure of N_2, a $\mathscr{T}(\mathfrak{A})$-closed set. Hence $x\in N_1\cap N_2$, a contradiction.

COROLLARY 1. *Let $(X,\ \mathscr{T})$ be connected. Let A be a dense subset of $(X,\ \mathscr{T})$. Then $(X,\ \mathscr{T}(A))$ is connected.*

COROLLARY 2. *Let $(X,\ \mathscr{T})$ be connected. Let A be a dense subset of $U\in\mathscr{T}$. Then $(X,\ \mathscr{T}(A))$ is connected.*

PROOF. Extend $\mathscr{T}$ to the set $B=A\cup(X-\bar{U})$, which is dense in $(X,\ \mathscr{T})$. Apply Corollary 1, then observe that $\mathscr{T}(B)=\mathscr{T}(A)$.

It should be noted that it is not true that an extension of the topology of a connected space to an arbitrary collection of dense subsets preserves connectedness. To see this, extend the usual topology $\mathcal{T}$ of the real line to A_1, the set of rationals, and A_2, the set of irrationals. Notice that, whereas connectedness is preserved under a simple extension of $\mathcal{T}$ to either A_1 or A_2, the real line with the finite extension topology $\mathcal{T}(\{A_1, A_2\})$ is not connected, for the sets A_1 and A_2 disconnect the space.

4. Applications to Maximal Connectedness.

A maximally connected topology for a space is a connected topology which is not properly contained in any other connected topology. Such topologies have recently been investigated by Thomas [3] and others. Since a strictly finer topology can always be obtained by means of a simple extension to a non-open set, we have the following corollary to our main theorem:

COROLLARY 3. *In a maximally connected space, every dense set is open.*

Examples have been given in [1] and [3] to show that the usual topology of the real line is not maximally connected. This fact easily follows from Corollary 3 and the observation that the rationals are dense but not open.

It is not known whether the usual topology of the reals can be extended to a maximally connected topology. In fact, it is not even known whether there exists a maximally connected Hausdorff space. It is clear, however, from our Corollary 3, that in the search for such a space, one need consider only those spaces which do not contain a dense non-open subset.

West Virginia University
Morgantown, West Virginia

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