Kyungpook Math. J.
Volume 13, Number 2
December, 1973

ON SASAKIAN MANIFOLD

By Ram Hit

1. Introduction.

Let us consider an n-dimensional real differentiable manifold of differentiability class C^{r+1} endowed with a vector valued linear function F, a 1-form A, and a vector field T satisfying

$$\overline{\overline{X}}+X=A(X)T, \tag{1.1}$$

for arbitrary vector field X, where $\overline{X} = F(X)$. The system (F, T, A) is said to give an almost contact structure to M_n and M_n is called an almost contact manifold. From (1.1) we have

$$\text{rank } (F)=n-1,\ \overline{T}=0,\ A(\overline{X})=0, \quad A(T)=1,\ n \text{ is odd, say } 2m+1. \tag{1.2}$$

Agreement 1.1. The equations containing $X, Y, Z, \ldots$ etc. will hold for arbitrary vector fields $X, Y, Z, \ldots,$ etc.

Let there be defined in an almost contact manifold M_n, a metric tensor G satisfying

$$G(\overline{X}, \overline{Y})=G(X, Y)-A(X)A(Y). \tag{1.3}$$

Then the manifold M_n is called an almost contact metric manifold. Let us put

$$'F(X, Y)=G(\overline{X}, Y). \tag{1.4}$$

Then from (1.1), (1.2), (1.3) and (1.4) we have

$$'F(\overline{X}, \overline{Y})=-G(X, \overline{Y})=G(\overline{X}, Y)={}'F(X, Y) \quad 'F(X, Y)+{}'F(Y, X)=0. \tag{1.5}$$

If in an almost contact metric manifold

$$'F(X, Y)=(D_XA)(Y)-(D_YA)(X)=(dA)(X, Y), \tag{1.6}$$

where D is a Riemannian connexion, then M_n is called an almost Sasakian manifold. In an almost Sasakian manifold $'F$ is closed:

$$(D_X{}'F)(Y, Z)+(D_Y{}'F)(Z, X)+(D_Z{}'F)(X, Y)=0. \tag{1.7}$$

An almost Sasakian manifold is said to be a Sasakian manifold if T is a Killing vector:

$$(D_XA)(Y)+(D_YA)(X)=0. \tag{1.8}$$

Thus in a Sasakian manifold

(1.9) $$'F(X, Y)=(D_X A)(Y),$$
$$(D_X{}'F)(Y, Z)={}'K(X, Y, Z, T)$$

where $'K$ is the curvature tensor of type (0, 4) of M_n.

Also in a Sasakian manifold

(1.10)a $$'K(T, X, Y, T)=g(\overline{X}, \overline{Y})=g(X, Y)-A(X)A(Y),$$

(1.10)b $$'K(X, Y, Z, T)=A(K(X, Y, Z))=A(X)g(Y, Z)-A(Y)g(X, Z),$$

(1.10)c $$'K(T, X, Y, Z)=A(Z)g(X, Y)-g(X, Z)A(Y),$$

(1.11) $$\mathrm{Ric}(X, T)=2mA(X),$$

where Ric is the Ricci tensor.

2. Almost contact structures in almost contact manifolds.

Equation (1.1) is equivalent to

$$F(F(X))+X=A(X)T.$$

The above equation gives

(2.1) $$F(F(\mu(X)))+\mu(X)=A(\mu(X))T$$

where μ is a non-singular vector valued linear function.

Put

(2.2) $$\mu(F'(X))=\overline{\mu(X)}.$$

Using (2.2) in (2.1) we obtain

(2.3) $$\mu(F'(F'(X)))+\mu(X)=A(\mu(X))T.$$

This gives

$$'F(F'(X))+X=A(\mu(X))\mu^{-1}T,$$

or

(2.4) $$F'(F'(X))+X=A'(X)T',$$

where we have put

(2.5) $$A'(X)=A(\mu(X)),\ \mu^{-1}T=T'.$$

Thus we have

THEOREM 2.1. *If we define* F', A', T' *by* (2.2) *and* (2.5), *then* (F', T', A') *is also an almost contact structure.*

We have from (1.3)

$$G(\overline{\mu(X)}, \overline{\mu(Y)})=G(\mu(X), \mu(Y))-A(\mu(X))A(\mu(Y)),$$

or

$$G(\mu(F'(X)), \mu(F'(Y))=G(\mu(X), \mu(Y))-A(\mu(X))A(\mu(Y))$$

or

(2.6) $G'(F'(X),\ F'(Y))=G'(X,\ Y)-A'(X)A'(Y),$

where

$$G'(X,\ Y)=G(\mu(X),\ \mu(Y)).$$

Thus equations (2.4) and (2.6) define almost contact metric structure.

Let us put

(2.7) $'F'(X,\ Y)=G'(F'(X),\ Y).$

Then from (2.2), (2.6) and (2.7) we have

(2.8) $'F'(\mu(X),\ \mu(Y))='F'(X,\ Y).$

From (2.8) we have

$$\begin{aligned}(D_{\mu(X)}{}'F)(\mu(Y),\ \mu(Z))&=\mu(X)('F'(Y,\ Z))-'F(\mu(D_{\mu(X)}Y),\ \mu(Z))\\&-'F(\mu(Y),\ D_{\mu(X)}\mu(Z))-'F((D_{\mu(X)}\mu)(Y),\ \mu(Z))\\&-'F(\mu(Y),\ (D_{\mu(X)}\mu)(Z)),\\&=\mu(X)('F'(Y,\ Z))-'F'(D'_{\mu(X)}Y,\ Z)\\&-'F'(Y,\ D'_{\mu(X)}(Z)-'F((D_{\mu(X)}\mu)(Y),\ \mu(Z))\\&-'F(\mu(Y),\ (D_{\mu(X)}\mu)(Z))\\&=(D'_{\mu(X)}{}'F')(Y,\ Z)-'F((D_{\mu(X)}\mu)(Y),\ \mu(Z))\\&-'F(\mu(Y),\ (D_{\mu(X)}\mu)(Z)).\end{aligned}$$

From the above equation we have the following theorem.

THEOREM 2.2. *Let* $(F,\ T,\ A,\ G,\ D)$ *and* $(F',\ T',\ A',\ G',\ D')$ *be two almost contact metric structures related by* (2.2) *and* (2.4) *in an almost contact metric manifold. Let* $'F$ *be Killing with respect to* D. *Then* $'F'$ *is not Killing with respect to* D'.

3. Projective curvature tensor.

Weyl projective curvature tensor W is given by

(3.1) $W(X,\ Y,\ Z)=K(X,\ Y,\ Z)-\frac{1}{2m}\{X\ \mathrm{Ric}(Y,\ Z)-Y\ \mathrm{Ric}(X,\ Z)\}$

Let us put

$$'W(X,\ Y,\ Z,\ U)\ =\ g(W(X,\ Y,\ Z),\ U).$$

Then (3.1) becomes

(3.2) $'W(X,\ Y,\ Z,\ U)='K(X,\ Y,\ Z,\ U)-\frac{1}{2m}\{g(X,\ U)\ \mathrm{Ric}(Y,\ Z)-g(Y,\ U)\ \mathrm{Ric}(X,\ Z)\}.$

From (1.2), (1.10), (1.11) and (3.2), we have the following formulae in Sasakian manifold

(3.3) a) $'W(X,Y,Z,T)=A(W(X,Y,Z))=A(X)\{g(Y,\ Z)-\frac{1}{2m}\mathrm{Ric}(Y,Z)\}$

b) $'W(T, Y, Z, U)=A(U)\{g(Y, Z)-\frac{1}{2m}\text{Ric}(Y, Z)\}$,

c) $'W(T, Y, Z, T)=g(Y, Z)-\frac{1}{2m}\text{Ric}(Y, Z)$.

We also know

$'W(Z, U, V, Y)\neq{}'W(V, Y, Z, U)$ in general and so we define

$$(3.4)\quad 'W^*(X, Y, Z, U)={}'K(X, Y, Z, U)-\frac{1}{4m}\{g(X, U)\,\text{Ric}(Y, Z)-g(Y, U)\,\text{Ric}(X, Z)+\text{Ric}(X, U)g(Y, Z)-\text{Ric}(Y, U)g(X, Z)\},$$

so that

$$(3.5)\qquad 'W^*(X, Y, Z, U)={}'W^*(Z, U, X, Y)$$

and

$$'W^*_{ijkl}\,\omega^{ij}\,\omega^{kl}={}'W_{ijkl}\,\omega^{ij}\,\omega^{kl}$$

where $'W^*_{ijkl}$ and $'W_{ijkl}$ are components of $'W^*$ and $'W$ and ω^{ij} is a skew-symmetric tensor [1].

This tensor in Riemannian fourfold enables to extend the Pirani formalism of gravitational waves to the Einstein space [2]. We shall see that this tensor is closely related to projective curvature in Sasakian manifold.

LEMMA 3.1. *In a Sasakian manifold we have*

$$(3.6)\qquad 2'W^*(X, Y, Z, T)={}'W(X, Y, Z, T).$$

PROOF. Putting T for U in (3.4) and using (1.2), (1.10), (1.11) and (3.3) a) we obtain (3.6).

A Sasakian manifold is said to be projectively symmetric if

$$(3.7)\qquad (D_Y W)(X, Y, Z)=0$$

holds. From (3.7) we have

$$(3.8)\quad K(X, Y, W(Z, U, V))-W(K(X, Y, Z), U, V)-W(Z, K(X, Y, U), V)-W(Z, U, K(X, Y, V))=0$$

This equation implies

$$(3.9)\quad 'K(T, Y, W(Z, U, V)T)-{}'W(K(T, Y, Z), U, V, T)-{}'W(Z, K(T, Y, U), V, T)-{}'W(Z, U, K(T, Y, V), T)=0.$$

Similarly we have

$$(3.10)\quad 'K(T, Y, W^*(Z, U, V), T)-{}'W^*(K(T, Y, Z), U, V, T)-{}'W^*(Z, K(T, Y, U), V, T)-{}'W^*(Z, U, K(T, Y, V), T)=0$$

From (3.6), (3.9) and (3.10) we conclude the following

THEOREM 3.1. *In a Sasakian manifold if*

(3.11) a) $(D_Y W)(X, Z, U)=0$, b) $(D_Y W^*)(X, Z, U)=0$

holds, then $'W(X, Z, U, V)=2'W^*(X, Z, U, V)$.

Let the Sasakian manifold be M-projectively flat. Then $'W^*(X, Y, Z, U)=0$, which implies

$$(3.12)\quad 'K(X, Y, Z, U)=\frac{1}{4m}\{g(X, U)\,\mathrm{Ric}(Y, Z)-g(Y, U)\,\mathrm{Ric}(X, Z) + \mathrm{Ric}(X, U)g(Y, Z)-\mathrm{Ric}(Y, U)g(X, Z)\}.$$

Putting T for X and U in (3.12) and using (1.2), (1.10)a) and (1.11) we obtain

$$(3.13)\quad \mathrm{Ric}(Y, Z)=2m\,g(Y, Z).$$

Substituting (3.13) in (3.12) we obtain

$$K(X, Y, Z, U)=g(X, U)g(Y, Z)-g(Y, U)g(X, Z).$$

Since M-projectively manifold is Einstein manifold

$$(3.14)\quad \mathrm{Ric}(Y, Z)=\frac{R}{n}\,g(Y, Z).$$

Comparing (3.13) and (3.14) we obtain

$$R=2mn.$$

Thus we have

THEOREM 3.2. *A M-projectively flat Sasakian manifold is a manifold of constant Riemannian curvature and the curvature is* $2mn$.

Acknowledgement. The author is grateful to Prof. R.S. Mishra for his kind advices and encouragements. He also thanks deeply to Prof. S. Sasaki (Japan) who has helped him in more than one way.

Faculty of Science
Banaras Hindu University
Varanasi, India

REFERENCES

[1] S. Tanno, : *Curvature tensors and non-existence of Killing vectors*, Tensor, N.S., vol. 22, 387—394, 1971.

[2] G.P. Pokhariyal and R.S. Mishra: *Curvature tensors and their relativistic significance* (II), The Yokohama Mathematical Journal, vol. 19, 97—103, 1971.