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A NOTE ON $\aleph$-SPACES

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In [10] O'Meara has introduced a new class of topological spaces, called $\aleph$-spaces. Following him, a regular T_1 space with a σ-locally finite k-network is called an $\aleph$-space. A collection $\mathscr{B}$ of subsets of X is said to be a k-network for X if for each compact subset K of X and each open subset U of X containing K there exists a finite union R of members of $\mathscr{B}$ such that $K \subset R \subset U$. $\mathscr{B}$ is said to be a pseudobase if for each compact subset K of X and each open subset U of X containing K there is a $B \in \mathscr{B}$ such that $K \subset B \subset U$. $\mathscr{B}$ is said to be a network for X if for each $x \in X$ and each open subset U of X containing x, there is a $B \in \mathscr{B}$ such that $x \in B \subset U$. A space X with a countable pseudo-base is called an $\aleph_0$-space by Michael [7], whereas a space X with a closed σ-locally finite network is called a σ-space by Okuyama [13]. The class of all σ-spaces contains the class of all $\aleph_0$-spaces, and all subparacompact spaces (that is, spaces with the property that every open covering has a σ-discrete closed refinement).

In [10,11] properties of $\aleph$-spaces parallel to $\aleph_0$-spaces have been obtained. In the present note some sum theorems for $\aleph$-spaces have been given. It is also proved that the image of an $\aleph$-space under a perfect mapping is an $\aleph$-space.

In the end, we obtain a sufficient condition for an invertible space to be an $\aleph$-space. Simple extension due to Levine [12] has also been considered for $\aleph$-spaces.

We shall first prove the locally finite sum theorem for $\aleph$-spaces which states the following:

THEOREM 1. *If $\{F_\alpha : \alpha \in \Omega\}$ is a locally finite closed covering of X such that each F_α is an $\aleph$-space, then X is an $\aleph$-space.*

PROOF. Since each F_α is a regular T_1 space, therefore X is a regular T_1 space. Thus we shall only prove that X has a σ-locally finite k-network if each F_α

has a σ-locally finite k-network. For each $\alpha \in \Omega$ let $\mathscr{V}^{\alpha}=\bigcup_{n=1}^{\infty}\mathscr{V}_n^{\alpha}$ be a σ-locally finite k-network for F_{α}, where each $\mathscr{V}_n^{\alpha}$ is locally finite in F_{α} (and hence in X). Then $\mathscr{W}=\bigcup_{n=1}^{\infty}\mathscr{W}_n$, where $\mathscr{W}_n=\bigcup_{\alpha\in\Omega}\mathscr{V}_n^{\alpha}$ is a σ-locally finite k-network for X. For, let K be a compact subset of X and U an open subset of X such that $K \subset U$. Since every locally finite family is compact finite (that is, every compact subset intersects at most finitely many members of the family), therefore K intersects at most finitely many F_{α}'s say $F_{\alpha_1}, F_{\alpha_2}, \cdots\cdots, F_{\alpha_k}$. Thus $K=\bigcup_{i=1}^{k}(K \cap F_{\alpha_i})\subset U$. For each $i=1, 2, \cdots\cdots, k$, $K\cap F_{\alpha_i}$ is a compact subset of F_{α_i} which is contained in an open subset $U \cap F_{\alpha_i}$ of F_{α_i}. Let R_i be a finite union of members of $\mathscr{V}^{\alpha_i}$ such that $K \cap F_{\alpha_i} \subset R_i \subset U \cap F_{\alpha_i} \subset U$. Therefore, $\bigcup_{i=1}^{k} R_i$ is a finite union of members of $\mathscr{W}$ such that $K\subset\bigcup_{i=1}^{k} R_i \subset U$. Hence $\mathscr{W}$ is a σ-locally finite k-network.

COROLLARY 1. *Every disjoint topological sum of $\aleph$-spaces is an $\aleph$-space.*

It has been proved by Hodel [3] that for any topological property P which is closed hereditary (that is a property, which when possessed by a space, is possessed by every closed subset of it) and which satisfies the locally finite sum theorem, the following theorems are true.

THEOREM 2. *If $\mathscr{V}$ is a σ-locally finite open covering of a space X such that the closure of each member of $\mathscr{V}$ has the property P, then X has the property P.*

THEOREM 3. *Let X be a regular topological space and let $\mathscr{V}$ be a σ-locally finite open covering of X such that each member of $\mathscr{V}$ has the property P and the frontier of each member of $\mathscr{V}$ is compact. Then X has the property P.*

THEOREM 4. *If $\mathscr{V}$ is a σ-locally finite elementary covering of X such that each member of $\mathscr{V}$ has the property P, then X has the property P. (For the definition of elementary covering see definition* 1).

DEFINITION 1. [Hodel, 3]. A subset A of X is said to be *elementary* if it is open and if there exists a sequence $\{A_i\}_{i=1}^{\infty}$ of open subsets of X such that $A \subset \bigcup_{i=1}^{\infty} A_i$ and $\overline{A}_i \subset A$ for all i. A covering consisting of elementary sets is said to be an *elementary covering*.

DEFINITION 2. [Y. Katuta, 4]. A family $\{A_\alpha : \alpha \in \Omega\}$ of subsets of X is said to be *order locally finite* if there is a linear ordering '$<$' of the index set Ω such that for each $\alpha \in \Omega$, the family $\{A_\beta : \beta < \alpha\}$ is locally finite at each point of A_α.

Every σ-locally finite family is order locally finite, but not conversely.

In [13], Singal and Arya have proved some sum theorems for order locally finite open coverings of X. Let P be a topological property which is closed hereditary and which satisfies the locally finite sum theorem, then the following two theorems hold.

THEOREM 5. *Let $\mathscr{V}$ be an order locally finite open covering of X such that the closure of each member of $\mathscr{V}$ possesses the property P. Then X possesses P.*

THEOREM 6. *If $\mathscr{V}$ is an order locally finite open covering of a regular space X such that each member of $\mathscr{V}$ possesses the property P and the frontier of each member of $\mathscr{V}$ is compact, then X has the property P.*

Obviously Theorems 2 and 3 of Hodel follow as corollaries to Theorems 5 and 6 respectively.

Since the property of being an ℵ-space is hereditary, therefore, in view of Theorem 1, we have the following theorems.

THEOREM 7. *If $\mathscr{V}$ is a σ-locally finite elementary covering of X such that each $V \in \mathscr{V}$ is an ℵ-space, then X is an ℵ-space.*

THEOREM 8. *If $\mathscr{V}$ is an order locally finite open covering of X such that the closure of each member of $\mathscr{V}$ is an ℵ-space, then X is an ℵ-space.*

THEOREM 9. *If $\mathscr{V}$ is an order locally finite open covering of a regular space X such that each member of $\mathscr{V}$ is an ℵ-space and frontier of each member of $\mathscr{V}$ is compact, then X is an ℵ-space.*

As a consequence of the locally finite sum theorem and the closed hereditary character of ℵ-spaces, we deduce the following interesting results.

THEOREM 10. *Let $\mathscr{V}$ be a locally finite open covering of a regular space X such that each member of $\mathscr{V}$ is an ℵ-space and frontier of each member of $\mathscr{V}$ is Lindelöf. Then X is an ℵ-space.*

PROOF. Let $\mathscr{V} = \{V_{\vec{\alpha}} : \alpha \in \Omega\}$ be the given locally finite open covering of X. For each $\alpha \in \Omega$, $\mathrm{Fr}\, V_\alpha$ is Lindelöf. Therefore there exists a countable subfamily $\{V_{\alpha_i} : i = 1, 2, \cdots\}$ of $\mathscr{V}$ which covers $\mathrm{Fr}\, V_\alpha$. Let $F_1 = \mathrm{Fr}\, V_\alpha \sim \bigcup_{i=2}^{\infty} V_{\alpha_i}$. Then F_1 is a closed subset of Fr V_α (and hence of X) such that $F_1 \subset V_{\alpha_1}$. Since F_1 is Lindelöf and X is regular there exists an open set U_1 such that $F_1 \subset U_1 \subset \overline{U}_1 \subset V_{\alpha_1}$. $\overline{U}_1$, being a closed subset of an $\aleph$-space V_{α_1}, is an $\aleph$-space. Suppose for each $i=1, 2, \cdots, n-1$, there exists an open set U_i such that $F_i \subset U_i \subset \overline{U}_i \subset V_{\alpha_i}$ where $F_i = \text{Fr } V_\alpha \sim [(\bigcup_{k=1}^{i-1} U_k) \cup (\bigcup_{k=i+1}^{\infty} V_\alpha)]$ and $\overline{U}_i$ is an $\aleph$-space. Now let,

$$F_n = \text{Fr } V_\alpha \sim [(\bigcup_{k=1}^{n-1} U_k) \cup (\bigcup_{k=n+1}^{\infty} V_\alpha)].$$

Then F_n is a closed Lindelöf subset such that $F_n \subset V_{\alpha_n}$. Again, by regularity of X there exists an open set U_n such that $F_n \subset U_n \subset \overline{U}_n \subset V_{\alpha_n}$ and $\overline{U}_n$ is an $\aleph$-space. Thus by induction we obtain a family $\mathscr{U} = \{U_n : n \in N\}$ of open sets satisfying:

(a) $\mathscr{U}$ is a covering of Fr V_α,

(b) $\{\overline{U}_n ; n \in N\}$ is locally finite.

Let $F_0 = \overline{V}_\alpha \sim \bigcup_{k=1}^{\infty} U_k$, then $\{\overline{U}_n ; n \in N\} \cup \{F_0\}$ is a locally finite closed covering of $\overline{V}_\alpha$ each member of which is an $\aleph$-space. Hence by Theorem 1, $\overline{V}_\alpha$ is an $\aleph$-space. Thus $\{\overline{V}_\alpha : \alpha \in \Omega\}$ is a locally finite closed covering of X each member of which is an $\aleph$-space. Hence X is an $\aleph$-space, in view of Theorem 1. For details of the proof, see [14].

THEOREM 11. *If $\mathscr{V}$ be a locally finite open covering of a normal space X such that each $V \in \mathscr{V}$ is an $\aleph$-space, then X is an $\aleph$-space.*

PROOF. Let $\mathscr{V} = \{V_\alpha : \alpha \in \Omega\}$. Since X is normal, there exists an open covering $\{U_\alpha : \alpha \in \Omega\}$ of X such that $\overline{U}_\alpha \subset V_\alpha$. Then $\{\overline{U}_\alpha : \alpha \in \Omega\}$ is a locally finite closed covering of X such that each $\overline{U}_\alpha$ is an $\aleph$-space. Hence X is an $\aleph$-space.

An open covering $\mathscr{V}$ of X is said to be a normal open covering if there is a sequence $\{\mathscr{V}_n\}$ of open coverings of X such that each $\mathscr{V}_n$ is a star-refinement of $\mathscr{V}_{n-1}$ and $\mathscr{V}_1$ is a refinement of $\mathscr{V}$.

THEOREM 12. *Let $\mathscr{V}$ be a normal open covering of a normal space X. Then X is an $\aleph$-space if each $V \in \mathscr{V}$ is an $\aleph$-space.*

PROOF. Since $\mathscr{V}$ is a normal open covering of the normal space X, therefore $\mathscr{V}$ admits of a locally finite open refinement [8, Theorem 1.2.]. Hence the result follows in view of Theorem 11.

THEOREM 13. *If $\mathscr{V}$ be a point finite open covering of a collectionwise normal space such that each member of $\mathscr{V}$ is an $\aleph$-space, then X is an $\aleph$-space.*

PROOF. The result follows in view of Theorem 11 and the fact that in a collectionwise normal space, every point finite open covering has a locally finite open refinement [6].

THEOREM 14. *Let $\mathscr{V}$ be a σ-locally finite open covering of a normal space X such that each $V \in \mathscr{V}$ is an F_σ-subset of X. If each $V \in \mathscr{V}$ is an $\aleph$-space, then X is an $\aleph$-space.*

PROOF. By Theorem 1.2 in [8], $\mathscr{V}$ is a normal covering. Hence the result follows in view of Theorem 12.

THEOREM 15. *Let $\mathscr{V}$ be a σ-locally finite open covering of a countably paracompact normal space X such that each member of $\mathscr{V}$ is an $\aleph$-space. Then X is an $\aleph$-space.*

PROOF. Since every σ-locally finite open covering of a countably paracompact normal space is normal (see [9]), the result follows in view of Theorem 12 above.

THEOREM 16. *Let X be a regular space which is the union of two sets A and B such that A is nonempty compact and B is paracompact. If $\mathscr{V}$ be an open covering of X such that each $V \in \mathscr{V}$ is an $\aleph$-space, then X is an $\aleph$-space.*

PROOF. Let $\mathscr{V} = \{V_\alpha : \alpha \in \Omega\}$. For each $x \in A$ there is an $\alpha_x \in \Omega$ such that $x \in V_{\alpha_x}$. Since X is regular, let U_{α_x} be an open subset of X such that $x \in U_{\alpha_x} \subset \overline{U}_{\alpha_x} \subset V_{\alpha_x}$. Let $\{U_{\alpha_{x_1}}, \cdots\cdots, U_{\alpha_{x_n}}\}$ be a finite subfamily of $\{U_{\alpha_x} : x \in A\}$ such that $A \subset \bigcup_{i=1}^{n} U_{\alpha_{x_i}}$. Also, each $\overline{U}_{\alpha_{x_i}}$, being a subset of $V_{\alpha_{x_i}}$ is an $\aleph$-space. Let $F = X \sim \bigcup_{i=1}^{n} U_{\alpha_{x_i}}$. Then F is a closed subset of X which is contained in B and hence F is a regular paracompact space. Therefore the covering $\{F \cap V_\alpha : \alpha \in \Omega\}$ of F has a locally finite (in F and hence in X) closed (in F and hence in X) refinement $\mathscr{U}$. The covering $\mathscr{W} = \{U : U \in \mathscr{U}\} \cup \{\overline{U}_{\alpha_{x_i}} : i = 1, 2, \cdots, n\}$ is then a locally finite closed covering of X such that each $W \in \mathscr{W}$ is an $\aleph$-space. Hence X is an $\aleph$-space in view of Theorem 1.

THEOREM 17. *Let X be a collectionwise normal space and let X be the union of two sets A and B such that A is paracompact and closed and B is paracompact. If $\mathscr{V}$ is an open covering of X such that each $V \in \mathscr{V}$ is an $\aleph$-space then X is an*

$\aleph$-space.

PROOF. Let $\mathscr{V}=\{V_\alpha: \alpha\in\Omega\}$. Since A is paracompact, the open covering $\{A\cap V_\alpha: \alpha\in\Omega\}$ of A has a locally finite open refinement $\{U_\beta: \beta\in\Gamma\}$. Since X is collectionwise normal, therefore by Lemma 1 in [1] there exists a locally finite open covering $\{W_\beta: \beta\in\Gamma\}$ of X such that $A\cap W_\beta\subset U_\beta$ for each $\beta\in\Gamma$. For each $\beta\in\Gamma$, let $\alpha(\beta)\in\Omega$ such that $U_\beta\subset A\cap V_{\alpha(\beta)}$. Let $G_\beta=W_\beta\cap V_{\alpha(\beta)}$. Then $\mathscr{G}=\{G_\beta: \beta\in\Gamma\}$ is a locally finite open collection which covers A. Since X is regular, there exists a locally finite open collection $\mathscr{H}'=\{H_\delta: \delta\in\Delta\}$ which covers A and is such that each H_δ is contained in some G_β. Let $F=X\sim\bigcup\{H_\delta: \delta\in\Delta\}$. Then F is a closed subset of X which is contained in B, and hence is paracompact. As above, we obtain a locally finite closed collection $\mathscr{H}''$ which covers F and such that each member of $\mathscr{H}''$ is an $\aleph$-space. Thus $\mathscr{H}=\{H_\delta: \delta\in\Delta\}\cup\mathscr{H}''$ is a locally finite closed covering of X such shat each member of $\mathscr{H}$ is an $\aleph$-space. Hence X is an $\aleph$-space.

THEOREM 18. *Let $\mathscr{F}=\{F_n: n=1,2,\cdots\}$ be a countable family of closed subsets of X such that $\bigcup\{F_n^{\circ}: n=1,2,\cdots\}=X$. If each F_n is an $\aleph$-space, then X is an $\aleph$-space.*

PROOF. Since F_n is a closed subset, therefore $F_n^{\circ -}$ is contained in F_n and hence is an $\aleph$-space. Thus $\{F_n^{\circ}: n=1,2,\cdots\}$ is a σ-locally finite open covering of X such that each $F_n^{\circ -}$ is an $\aleph$-space. Hence the result follows in view of Theorem 2.

DEFINITION 3. [Frolik, 2]. A space X is said to be *weakly regular* if every non-empty open subset of X contains a non-empty regularly closed set.

THEOREM 19. *Every space which contains a proper, nonempty regularly closed subset is an $\aleph$-space if and only if every regularly closed subset of X is an $\aleph$-space.*

PROOF. The 'only if' part is obvious. We shall, therefore, prove the 'if' part. Let X be a space containing a proper nonempty regularly closed set U. Therefore $U=U^{\circ -}$. Let $U^{\circ}=V$. Then V is contained in U and so $\overline{V}$ is a proper regularly closed subset of X where V is regularly open. Thus $X=\overline{V}\cup(X\sim V)$, where V and $X\sim V$ are both $\aleph$-spaces. Hence X is an $\aleph$-space.

COROLLARY 2. *A weakly regular space X is an $\aleph$-space if and only if every proper regularly closed subset of X is an $\aleph$-space.*

COROLLARY 3. *A semi-regular space X is an $\aleph$-space if and only if every proper regularly closed subset of X is an $\aleph$-space.*

A mapping f: $X \to Y$ is called a perfect mapping if it is closed, continuous and such that $f^{-1}(y)$ is compact for each $y \in Y$.

THEOREM 22. Let $f: X \to Y$ be a perfect mapping. Then Y is an ℵ-space if X is so.

PROOF. Let X be an ℵ -space and let $\mathscr{V} = \bigcup_{n=1}^{\infty} \mathscr{V}_n$ be a σ-locally finite k-network for X. We shall prove that $\mathscr{W} = \bigcup_{n=1}^{\infty} \mathscr{W}_n$, where $\mathscr{W}_n = \{f(V): V \in \mathscr{V}_n\}$ is a σ-locally finite k-network for Y. Since f is continuous, each $\mathscr{W}_n$ will be locally finite in Y. To prove that $\mathscr{W}$ is a k-network for Y, let K be a compact subset of Y and U be an open subset of Y such that $K \subset U$. Since f is a closed continuous mapping with $f^{-1}(y)$ compact for each $y \in Y$, therefore $f^{-1}(K)$ is a compact subset of X contained in the open set $f^{-1}(U)$. Let R be a finite union of members of such that $f^{-1}(K) \subset R \subset f^{-1}(U)$. Thus $K \subset f(R) \subset U$ and $f(R)$ is a finite union of members of $\mathscr{W}$. Hence Y is an ℵ-space, since it is regular also as X regular.

DEFINITION 4. [Doyle and Hocking, 2]. A space X is said to be an *invertible space* if for each open subset U of X there is a homeomorphism h: $X \to X$ such that $h(X-U) \subset U$. h is called an *inverting homeomorphism* for U.

THEOREM 23. *Let X be a topological space invertible in one of its non-empty open subsets U and let $\bar{U}$ be an ℵ-space, then X is an ℵ-space.*

PROOF. Let f be an inverting homeomorphism for U. Then $f(\bar{U})$ is closed and $X = \bar{U} \cup f(\bar{U})$. Since $\bar{U}$ and $f(\bar{U})$ are ℵ-spaces, therefore by Theorem 1, X is an ℵ-space.

DEFINITION 5. [Levine, 12]. Let (X, τ) be any topological space. Then the topology $\tau(A) = \{U \cup (V \cap A): U, V \in \tau\}$ where $A \notin \tau$, is called a *simple extension* for τ. Obviously $A \in \tau(A)$. As is easily verified, $(A, \tau \cap A) = (A, \tau(A) \cap A)$ and $(X-A, \tau \cap (X-A)) = (X-A, \tau(A) \cap (X-A))$.

THEOREM 24. *Let (X, τ) be an ℵ-space and A be a closed subspace of (X, τ). Then $(X, \tau(A))$ is an ℵ-space.*

PROOF. Since (X, τ) is an ℵ-space, therefore $(A, \tau \cap A)$ and $(X-A, \tau \cap (X-A))$ are ℵ-space. But $(A, \tau \cap A) = (A, \tau(A) \cap A)$ and $(X-A, \tau(A) \cap (X-A)) = ((X-A), \tau \cap (X-A))$. Thus X is the union of two $\tau(A)$-closed ℵ-spaces A and $X-A$. Hence $(X, \tau(A))$ is an ℵ-space.

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