Kyungpook Math. J.
Volume 13, Number 2
December, 1973

POINCARÉ PERIOD RELATION ON COMPACT RIEMANN SURFACES

By Ok Young Yoon

1. Introduction.

Let S be a compact Riemann surface of genus $g \geq 2$, $\Gamma=(\gamma_1, \gamma_2, \cdots, \gamma_g)$; $\Delta=(\delta_1, \delta_2, \cdots, \delta_g)$ (abbreviated by (Γ, Δ)) a canonical homology basis on S and du_1, du_2, $\cdots$, du_g a normalized abelian differentials of the first kind on S, uniquely determined by the given homology basis (Γ, Δ), such that

(1) $$\int_{\gamma_j} du_i=\delta_{ij}, \ \int_{\delta_j} du_i=\pi_{ij}, \ i,j=1,2,\cdots,g.$$

The $g \times g$ matrix $\Pi=(\pi_{ij})$ is said to be the period matrix (by contrast, $g \times 2g$ matrix $(1_g, \Pi)$, where 1_g is the $g \times g$ identity matrix, the full period matrix) of S and (Γ, Δ).

It is well known that Π is complex symmetric with positive definite imaginary part, the set of all such matrices is generally called the Siegel (or generalized) upper-half plane $\mathfrak{S}_g$ of degree (or genus) g, and that are holomorphic functions of $3g-3$ complex parameters, "the moduli", for a non-hyperelliptic Riemann surface, of $2g-1$ complex parameters for a hyperelliptic Riemann surface, S of genus $g \geq 2$ [4]. Consequently, there are $\frac{(g-2)(g-3)}{2}$ holomorphic relations for a non-hyperelliptic Riemann surface S of genus $g \geq 4$ and $\frac{(g-1)(g-2)}{2}$ holomorphic relations for a hyperelleptic Riemann surface S of genus $g \geq 3$ among π_{ij}. Such holomorphic relations of π_{ij} are usually called the period relations on compact Riemann surfaces. One of the classical problems in the theory of compact Riemann surfaces is to formulate such relations, for more than a centry since Riemann.

In 1888 [5,6], Schottky first derived a period relation for $g=4$ and 5

(2) $$\sqrt{r_1} \pm \sqrt{r_2} \pm \sqrt{r_3}=0,$$

where $r_i, i=1,2,3,$ are the products of 8 theta contants associated with S and a definite canonical homology basis (Γ, Δ).

On the other hand, in 1895 [2], Poincaré obtained an approximate period relation

(3) $$\sqrt{\pi_{13}\pi_{14}\pi_{23}\pi_{24}} \pm\sqrt{\pi_{12}\pi_{14}\pi_{32}\pi_{34}} \pm\sqrt{\pi_{12}\pi_{13}\pi_{42}\pi_{43}} = 0$$

for compact Riemann surfaces of genus $g=4$ whose period matrices Π's are close to diagonal form. Recently, Rauch [3] showed that Schottky's period relation implies Poincaré's approximate period relation for compact Riemann surfaces of genus $g=4$, and I [7] showed that a period reation of Schottky type $\sqrt{r_1}\pm\sqrt{r_2}\pm\sqrt{r_3}=0$ holds on hyperelliptic Riemann surfaces of genus $g\geq 4$, which can be recognized as a generalization to a relation for genus $g>4$ from a relation for genus $g=4$.

In the present paper, on hyperelliptic Riemann surfaces of genus $g\geq 4$, we shall obtain a period relation of Schottky type

(4) $$\sqrt{r_1}\pm\sqrt{r_2}\pm\sqrt{r_3}=0,$$

where

(5) $$r_1=\theta\begin{bmatrix}10100\cdots0\\10100\cdots0\end{bmatrix}\theta\begin{bmatrix}10010\cdots0\\11110\cdots0\end{bmatrix}\theta\begin{bmatrix}01100\cdots0\\01110\cdots0\end{bmatrix}\theta\begin{bmatrix}11010\cdots0\\01010\cdots0\end{bmatrix}\times$$
$$\theta\begin{bmatrix}00010\cdots0\\10000\cdots0\end{bmatrix}\theta\begin{bmatrix}00100\cdots0\\11010\cdots0\end{bmatrix}\theta\begin{bmatrix}11100\cdots0\\00000\cdots0\end{bmatrix}\theta\begin{bmatrix}01010\cdots0\\00100\cdots0\end{bmatrix},$$
$$r_2=\theta\begin{bmatrix}11000\cdots0\\11000\cdots0\end{bmatrix}\theta\begin{bmatrix}11110\cdots0\\10010\cdots0\end{bmatrix}\theta\begin{bmatrix}01110\cdots0\\11100\cdots0\end{bmatrix}\theta\begin{bmatrix}10110\cdots0\\00110\cdots0\end{bmatrix}\times$$
$$\theta\begin{bmatrix}01000\cdots0\\10110\cdots0\end{bmatrix}\theta\begin{bmatrix}10000\cdots0\\01100\cdots0\end{bmatrix}\theta\begin{bmatrix}00000\cdots0\\00010\cdots0\end{bmatrix}\theta\begin{bmatrix}00110\cdots0\\01000\cdots0\end{bmatrix},$$
$$r_3=\theta\begin{bmatrix}11000\cdots0\\11110\cdots0\end{bmatrix}\theta\begin{bmatrix}11110\cdots0\\10100\cdots0\end{bmatrix}\theta\begin{bmatrix}01110\cdots0\\11010\cdots0\end{bmatrix}\theta\begin{bmatrix}00110\cdots0\\01110\cdots0\end{bmatrix}\times$$
$$\theta\begin{bmatrix}01000\cdots0\\10000\cdots0\end{bmatrix}\theta\begin{bmatrix}10000\cdots0\\01010\cdots0\end{bmatrix}\theta\begin{bmatrix}00000\cdots0\\00100\cdots0\end{bmatrix}\theta\begin{bmatrix}10110\cdots0\\00000\cdots0\end{bmatrix},$$

and then derive a Poincaré's approximate period relation

(6) $$\sqrt{\pi_{13}\pi_{14}\pi_{23}\pi_{24}+0(\varepsilon^{10})}\pm\sqrt{\pi_{12}\pi_{14}\pi_{32}\pi_{34}+0(\varepsilon^{10})}\pm\sqrt{\pi_{12}\pi_{13}\pi_{42}\pi_{43}+0(\varepsilon^{10})}=0,$$

where $\varepsilon=\max_{1\leq i<j\leq g}\{\sqrt{|\pi_{ij}|}\}$, from (4) in case the period matrices are close to diagonal form.

In Section 2 we review some of well-known facts about compact Riemann surfaces and Riemann theta functions.

In Section 3 we obtain (4) on a hyperelliptic Riemann surface S of genus $g\geq 4$, choosing suitably a canonical homology basis (Γ, Δ) and considering two kinds of multiplicative functions on S.

In Section 4 we give the precise computations to derive (5) from (4) on S in case the period matrix Π of S and (Γ, Δ) is close to diagonal form.

2. Terminologies and Notations.

Throughout this paper we primarily concern for a compact Riemann surface S of genus $g\geq 4$, but assume for the moment $g\geq 1$.

Let C be the field of complex numbers. 2_g columns $e^{(j)}$, $\pi^{(j)}$, $j=1,2,\cdots,g$, of the full period matrix $(1_g,\Pi)$ of S and (Γ,Δ) are linearly independent over the reals R and generates a discrete abelian subgroup L of C^g of g complex variables. The quotient group C^g/L is called the Jacobi variety $J(S)$ of S and $J(S)$ is a compact abelian group.

An integral linear combination of $e^{(j)}$, $\pi^{(j)}$, $j=1,2,\cdots,g$, i.e.,

$$(7)\qquad \begin{Bmatrix}\varepsilon\\ \varepsilon'\end{Bmatrix}=\begin{Bmatrix}\varepsilon_1\cdots\cdots\varepsilon_g\\ \varepsilon_1'\cdots\cdots\varepsilon_g'\end{Bmatrix}=\sum_{j=1}^{g}\varepsilon_j'e^{(j)}+\sum_{j=1}^{g}\varepsilon_j\pi^{(j)},$$

is called a period in $J(S)$. The half of a period,

$$(8)\qquad \begin{pmatrix}\varepsilon\\ \varepsilon'\end{pmatrix}=\frac{1}{2}\begin{Bmatrix}\varepsilon\\ \varepsilon'\end{Bmatrix},$$

is called a half period in $J(S)$.

Fixing a point P_0 as a base point on S, we can define a map $u: S\longrightarrow J(S)$ by $u(P)=\left(\int_{P_0}^{P}du_1,\cdots,\int_{P_0}^{P}du_g\right)$ for each point P on S. Since a change of a path of integration from P_0 to P differs by a period, u has a well defined image in $J(S)$. Furthermore, this map u can be extented to an arbitrary divisor $S=P_1^{n_1}\cdots P_m^{n_m}$ on S by

$$(9)\qquad u(\zeta)=(u_i(\zeta))=\left(\sum_{j=1}^{m}n_j\int_{P_0}^{P_j}du_i\right),$$

where $i=1,2,\cdots,g$ and $n_j, j=1,2,\cdots,m$, are integers.

$2\times g$ matrix $\begin{bmatrix}\mu\\ \mu'\end{bmatrix}=\begin{bmatrix}\mu_1\cdots\cdots\mu_g\\ \mu'_1\cdots\cdots\mu'_g\end{bmatrix}$, where μ_j, μ'_j, $j=1,2,\cdots,g$, are 0 or 1, i. e., in Z_2, is called a characteristic.

$\begin{bmatrix}\mu\\ \mu'\end{bmatrix}$ is called even or odd depending on whether $\mu\cdot\mu'=\sum_{j=1}^{g}\mu_j\mu_j'\equiv 0$ or $1 (\mathrm{mod.}\ 2)$. There are $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd characteristics.

A meromorphic multi-valued function f on S is called multiplicative with characteristic $\begin{bmatrix}\nu\\ \nu'\end{bmatrix}=\begin{bmatrix}\nu_1\cdots\cdots\nu_g\\ \nu'_1\cdots\cdots\nu'_g\end{bmatrix}$, where ν_j, ν'_j $j=1, 2, \cdots, g$, are 0 or 1, if a continuati[illegible] of f along γ_j(or δ_j) carries to $(-1)^{\nu_j} f$ (or $(-1)^{\nu'_j} f$).

For $u=$[illegible]$, \cdots, u_g)$ in $C^g, T=(t_{ij})$ in $\mathfrak{S}_g$ and a characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$, the function defined by

(10) $$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u, T)=\sum_{N\in Z^g}\exp 2\pi i\left[\frac{1}{2}\left(N+\frac{\varepsilon}{2}\right)T\left(N+\frac{\varepsilon}{2}\right)+\left(N+\frac{\varepsilon}{2}\right)\left(u+\frac{\varepsilon'}{2}\right)\right]$$

is called (the first order) theta function with characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ and theta matrix T. The theta constant with characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ at T is

(11) $$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0, T)=\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}.$$

$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,T)$ converges absolutely and uniformly on compact subsets of $C^g\times\mathfrak{S}_g$, and hence it is an analytic function on $C^g\times\mathfrak{S}_g$. We note that the theta constant $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(0, T)$ is an analytic function of T. The following properties of theta functions will be useful in the future;

(12) Functional equation; $$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\left(u+\begin{Bmatrix}\mu\\ \mu'\end{Bmatrix}, T\right)$$
$$=\exp\pi i\,[(\varepsilon\cdot\mu'-\varepsilon'\cdot\mu)-2\mu\cdot u-\mu T\mu]\;\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u, T),$$

(13) Reduction formula; $$\theta\begin{bmatrix}\varepsilon+2\mu\\ \varepsilon'+2\mu'\end{bmatrix}(u,T)=(-1)^{\varepsilon\cdot\mu'}\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,T),$$

(14) $$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(-u,T)=(-1)^{\varepsilon\cdot\varepsilon'}\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u,T),$$

(15) Substitution formula ; $$\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}\left(u+\begin{pmatrix}\mu\\ \mu'\end{pmatrix}, T\right)$$
$$=\exp\pi i\left[-\frac{1}{4}\mu T\mu-\frac{1}{2}\mu\cdot(\varepsilon'+\mu')-\mu\cdot u\right]\theta\begin{bmatrix}\varepsilon+\mu\\ \varepsilon'+\mu'\end{bmatrix}(u,T).$$

By (14), $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u, T)$ is even or odd function of u depending on whether $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ is even or odd.

Hence there are $2^{g-1}(2^g+1)$ even and $2^{g-1}(2^g-1)$ odd theta functions of u, and theta constants $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ with odd characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ always vanish, but not $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ with even characteristic $\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}$ necessarily.

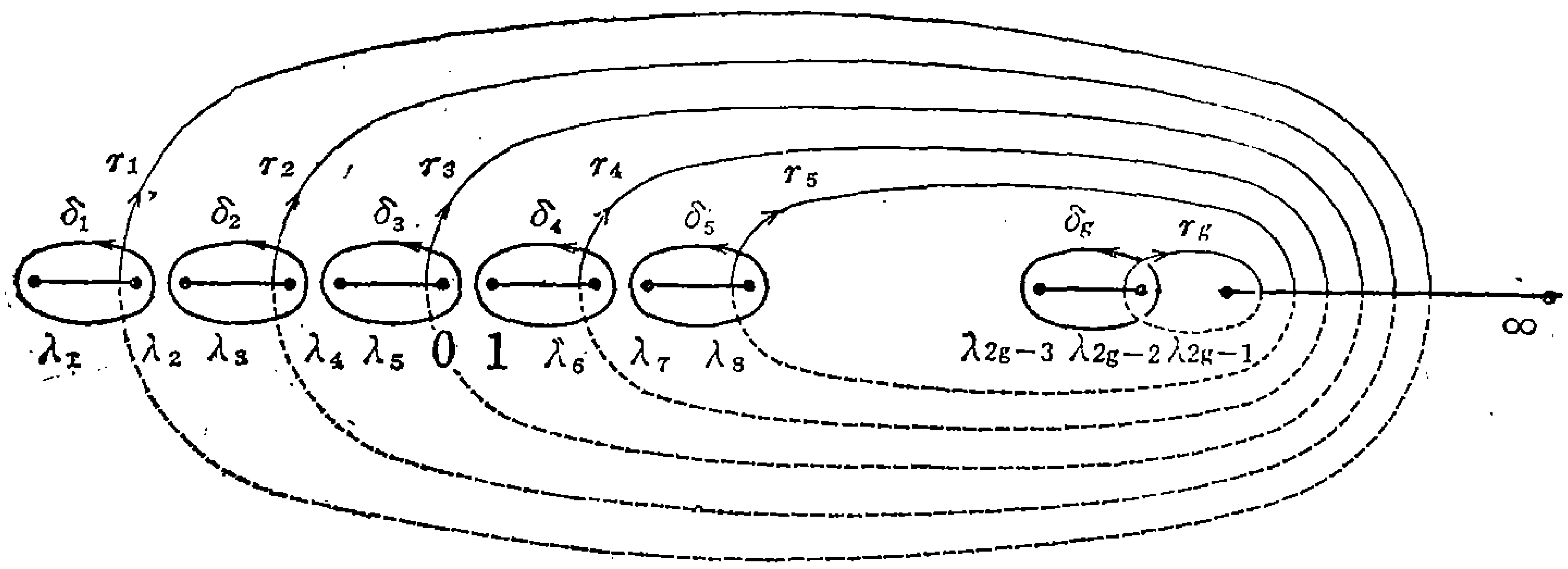
Since the period matrix Π of a compact Riemann surface S of genus $g\geq 1$ with a canonical homology basis (Γ, Δ) is in $\mathfrak{S}_g$, we can have theta functions defined by $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u(P), \Pi)$ on S, choosing a base point P_0 on S and replacing u by $u(P)$ and T by Π in(10). These functions are called Riemann theta functions associated with S and (Γ, Δ). If $\theta\begin{bmatrix}\varepsilon\\ \varepsilon'\end{bmatrix}(u(P), \Pi)$ does not identically vanish on S, it has well defined and uniquely determined g zeros $\zeta=P_1\cdots P_g$ on S such that $u(\zeta)+K\equiv\begin{pmatrix}\varepsilon\\ \varepsilon'\end{pmatrix}$, where K is a vector of Riemann constants depending on a base point P_0

on S and (Γ, Δ). Furthermore we can easily check that the quotient $\dfrac{\theta\begin{bmatrix}\varepsilon \\ \varepsilon'\end{bmatrix}(u(P), \Pi)}{\theta\begin{bmatrix}\mu \\ \mu'\end{bmatrix}(u(P), \Pi)}$ of two Riemann theta functions on S is multiplicative function with characteristic $\begin{bmatrix}\varepsilon+\mu \\ \varepsilon'+\mu'\end{bmatrix}$ by the functional equation (12). See [1] and [7] for further references.

3. Period relation of Schottky type.

Let S be a hyperelliptic Riemann surface of genus $g \geq 4$ which can be realized as the Riemann surface of an algebraic funcion w on S, satisfying an irreducible algebraic equation $w^2 = z(z-1)(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_{2g-1})$, where λ_j, $j=1, 2, \cdots, 2g-1$, are mutually distinct finite different from 0 and 1.

We choose a canonical homology basis (Γ, Δ) on S as follows;



Now, taking a point P_0 (with $z(P_0)=\lambda_1$) on S as a base point and finding the uniquely determined normalized abelian differentials du_i, $i=1, 2, \cdots, g$, with respect to the chosen (Γ, Δ) on S, we have a map $u: S \to J(S)$.

In particular, we can find all the images of $2g+2$ branch points under u in $J(S)$:

(16)
$$u(\lambda_1) \equiv \begin{pmatrix}00000\cdots0 \\ 00000\cdots0\end{pmatrix},$$
$$u(\lambda_2) \equiv \begin{pmatrix}10000\cdots0 \\ 00000\cdots0\end{pmatrix},$$
$$u(\lambda_3) \equiv \begin{pmatrix}10000\cdots0 \\ 11000\cdots0\end{pmatrix},$$
$$u(\lambda_4) \equiv \begin{pmatrix}11000\cdots0 \\ 11000\cdots0\end{pmatrix},$$
$$u(\lambda_5) \equiv \begin{pmatrix}11000\cdots0 \\ 10100\cdots0\end{pmatrix},$$
$$u(0) \equiv \begin{pmatrix}11100\cdots0 \\ 10100\cdots0\end{pmatrix},$$

$$u(1) \equiv \begin{pmatrix} 11100\cdots0 \\ 10010\cdots0 \end{pmatrix},$$

$$u(\lambda_6) \equiv \begin{pmatrix} 11110\cdots0 \\ 10010\cdots0 \end{pmatrix},$$

$$u(\lambda_7) \equiv \begin{pmatrix} 111100\cdots0 \\ 100010\cdots0 \end{pmatrix},$$

$$u(\lambda_8) \equiv \begin{pmatrix} 111110\cdots0 \\ 100010\cdots0 \end{pmatrix},$$

$$\vdots$$

$$u(\lambda_{2k-1}) \equiv \begin{pmatrix} 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{k-1} 00\cdots0 \\ 1 \phantom{\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{k-1}} 10\cdots0 \end{pmatrix},$$

$$u(\lambda_{2k}) \equiv \begin{pmatrix} 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{k-1} 10\cdots0 \\ 1 \phantom{\begin{pmatrix} 1 \\ 0 \end{pmatrix}^{k-1}} 10\cdots0 \end{pmatrix}, \qquad 3 \leq k \leq g-1,$$

$$\vdots$$

$$u(\lambda_{2g-1}) \equiv \begin{pmatrix} 11111\cdots\cdots\cdots1 \\ 10000\cdots\cdots\cdots0 \end{pmatrix},$$

$$u(\infty) \equiv \begin{pmatrix} 00000\cdots\cdots\cdots0 \\ 10000\cdots\cdots\cdots0 \end{pmatrix}.$$

A vector $K(\lambda_1)$ of Riemann constants with respect to a base point P_0 is given by

$$(17) \qquad K(\lambda_1) \equiv \begin{pmatrix} g & g-1 & g-2\cdots\cdots1 \\ g & 1 & 1\cdots\cdots\cdots1 \end{pmatrix} \equiv \sum_{k=1}^{g} u(\lambda_{2k-1}).$$

At this point we can find many non-zero Riemann theta functions associated with S and (Γ, Δ). In particular,

(18) Theta functions — Zeros

Theta functions	Zeros
$\theta\begin{bmatrix} 11110\cdots0 \\ 00010\cdots0 \end{bmatrix}(u(P), \Pi)$	$0, \lambda_1, \lambda_2, \lambda_4, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix} 00010\cdots0 \\ 00110\cdots0 \end{bmatrix}(u(P), \Pi)$	$\infty, \lambda_1, \lambda_2, \lambda_4, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix} 11000\cdots0 \\ 01000\cdots0 \end{bmatrix}(u(P), \Pi)$	$0, \lambda_1, \lambda_2, \lambda_6, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix} 00100\cdots0 \\ 01100\cdots0 \end{bmatrix}(u(P), \Pi)$	$\infty, \lambda_1, \lambda_2, \lambda_6, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix} 10000\cdots0 \\ 11100\cdots0 \end{bmatrix}(u(P), \Pi)$	$0, \lambda_1, \lambda_5, \lambda_6, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix} 01100\cdots0 \\ 11000\cdots0 \end{bmatrix}(u(P), \Pi)$	$\infty, \lambda_1, \lambda_5, \lambda_6, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix} 10110\cdots0 \\ 10110\cdots0 \end{bmatrix}(u(P), \Pi)$	$0, \lambda_1, \lambda_4 \lambda_5, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix} 01010\cdots0 \\ 10010\cdots0 \end{bmatrix}(u(P), \Pi)$	$\infty, \lambda_1, \lambda_4, \lambda_5, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix} 11110\cdots0 \\ 00100\cdots0 \end{bmatrix}(u(P), \Pi)$	$1, \lambda_1 \lambda_2, \lambda_4, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$

$\theta\begin{bmatrix}00010\cdots0\\00110\cdots0\end{bmatrix}(u(P),\ \Pi)$	$\infty, \lambda_1, \lambda_2, \lambda_4, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix}11000\cdots0\\01110\cdots0\end{bmatrix}(u(P),\ \Pi)$	$1, \lambda_1, \lambda_2, \lambda_6, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix}00100\cdots0\\01100\cdots0\end{bmatrix}(u(P),\ \Pi)$	$\infty, \lambda_1, \lambda_2, \lambda_6, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix}10000\cdots0\\11010\cdots0\end{bmatrix}(u(P),\ \Pi)$	$1, \lambda_1, \lambda_5, \lambda_6, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix}01100\cdots0\\11000\cdots0\end{bmatrix}(u(P),\ \Pi)$	$\infty, \lambda_1, \lambda_5, \lambda_6, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix}10110\cdots0\\10000\cdots0\end{bmatrix}(u(P),\ \Pi)$	$1, \lambda_1\lambda_4, \lambda_5, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$
$\theta\begin{bmatrix}01010\cdots0\\10010\cdots0\end{bmatrix}(u(P),\ \Pi)$	$\infty, \lambda_1, \lambda_4, \lambda_5, \lambda_8, \lambda_{10}, \cdots, \lambda_{2g-2},$

$$\frac{\theta\begin{bmatrix}11110\cdots0\\00010\cdots0\end{bmatrix}(u(P),\ \Pi)}{\theta\begin{bmatrix}00010\cdots0\\00110\cdots0\end{bmatrix}(u(P),\ \Pi)}$$ is multiplicative function on S with characteristic $\begin{bmatrix}11100\cdots0\\00100\cdots0\end{bmatrix}$, with a zero 0 and a pole ∞. On the other hand, we easily see that $\sqrt{z}$ is also multiplicative function on S with characteristic $\begin{bmatrix}11100\cdots0\\00100\cdots0\end{bmatrix}$, with a zero 0 and a pole ∞. Hence there exists a constant C such that

$$\text{(19)} \qquad \sqrt{Z}=C\frac{\theta\begin{bmatrix}11110\cdots0\\00010\cdots0\end{bmatrix}(u(P),\ \Pi)}{\theta\begin{bmatrix}00010\cdots0\\00110\cdots0\end{bmatrix}(u(P),\ \Pi)}.$$

Putting P with $z(P)=1$,

$$\text{(20)} \qquad C=\frac{\theta\begin{bmatrix}00010\cdots0\\00110\cdots0\end{bmatrix}\left(\begin{pmatrix}11100\cdots0\\10010\cdots0\end{pmatrix},\ \Pi\right)}{\theta\begin{bmatrix}11110\cdots0\\00010\cdots0\end{bmatrix}\left(\begin{pmatrix}11100\cdots0\\10010\cdots0\end{pmatrix},\ \Pi\right)}$$

$$=-i\frac{\theta\begin{bmatrix}11110\cdots0\\10100\cdots0\end{bmatrix}}{\theta\begin{bmatrix}00010\cdots0\\10000\cdots0\end{bmatrix}}$$

using (13) and (15). Consequently, we have from (19) and (20)

$$\text{(21)} \qquad \sqrt{\lambda_3}=-i\frac{\theta\begin{bmatrix}11110\cdots0\\10100\cdots0\end{bmatrix}\ \theta\begin{bmatrix}01110\cdots0\\11010\cdots0\end{bmatrix}}{\theta\begin{bmatrix}00010\cdots0\\10000\cdots0\end{bmatrix}\ \theta\begin{bmatrix}10010\cdots0\\11110\cdots0\end{bmatrix}},$$

again using (13), (15) and putting P with $z(P)=\lambda_3$ in (19).

Carrying out the similar computations for the rest of functions listed in (18), we obtain the following four different expressions of $\sqrt{\lambda_3}$ and $\sqrt{\lambda_3-1}$, respectively;

$$(22)\qquad \sqrt{\lambda_3}=-i\frac{\theta\begin{bmatrix}11110\cdots0\\10100\cdots0\end{bmatrix}\theta\begin{bmatrix}01110\cdots0\\11010\cdots0\end{bmatrix}}{\theta\begin{bmatrix}00010\cdots0\\10000\cdots0\end{bmatrix}\theta\begin{bmatrix}10010\cdots0\\11110\cdots0\end{bmatrix}}=i\frac{\theta\begin{bmatrix}11000\cdots0\\11110\cdots0\end{bmatrix}\theta\begin{bmatrix}01000\cdots0\\10000\cdots0\end{bmatrix}}{\theta\begin{bmatrix}00100\cdots0\\11010\cdots0\end{bmatrix}\theta\begin{bmatrix}10100\cdots0\\10100\cdots0\end{bmatrix}}$$

$$=-i\frac{\theta\begin{bmatrix}10000\cdots0\\01010\cdots0\end{bmatrix}\theta\begin{bmatrix}00000\cdots0\\00100\cdots0\end{bmatrix}}{\theta\begin{bmatrix}01100\cdots0\\01110\cdots0\end{bmatrix}\theta\begin{bmatrix}11100\cdots0\\00000\cdots0\end{bmatrix}}=i\frac{\theta\begin{bmatrix}10110\cdots0\\00000\cdots0\end{bmatrix}\theta\begin{bmatrix}00110\cdots0\\01110\cdots0\end{bmatrix}}{\theta\begin{bmatrix}01010\cdots0\\00100\cdots0\end{bmatrix}\theta\begin{bmatrix}11010\cdots0\\01010\cdots0\end{bmatrix}},$$

and

$$(23)\qquad \sqrt{\lambda_3-1}=-i\frac{\theta\begin{bmatrix}11110\cdots0\\10010\cdots0\end{bmatrix}\theta\begin{bmatrix}01110\cdots0\\11100\cdots0\end{bmatrix}}{\theta\begin{bmatrix}00010\cdots0\\10000\cdots0\end{bmatrix}\theta\begin{bmatrix}10010\cdots0\\11110\cdots0\end{bmatrix}}=i\frac{\theta\begin{bmatrix}11000\cdots0\\11000\cdots0\end{bmatrix}\theta\begin{bmatrix}01000\cdots0\\10110\cdots0\end{bmatrix}}{\theta\begin{bmatrix}00100\cdots0\\11010\cdots0\end{bmatrix}\theta\begin{bmatrix}10100\cdots0\\10100\cdots0\end{bmatrix}}$$

$$=-i\frac{\theta\begin{bmatrix}10000\cdots0\\01100\cdots0\end{bmatrix}\theta\begin{bmatrix}00000\cdots0\\00010\cdots0\end{bmatrix}}{\theta\begin{bmatrix}01100\cdots0\\01110\cdots0\end{bmatrix}\theta\begin{bmatrix}11100\cdots0\\00000\cdots0\end{bmatrix}}=i\frac{\theta\begin{bmatrix}10110\cdots0\\00110\cdots0\end{bmatrix}\theta\begin{bmatrix}00110\cdots0\\01000\cdots0\end{bmatrix}}{\theta\begin{bmatrix}01010\cdots0\\00100\cdots0\end{bmatrix}\theta\begin{bmatrix}11010\cdots0\\01010\cdots0\end{bmatrix}}.$$

To obtain (23), we only need to consider two kinds of multiplicative functions on S with characteristic $\begin{bmatrix}11100\cdots0\\00010\cdots0\end{bmatrix}$, with a zero 1 and a pole ∞, for example $\sqrt{z-1}$ and $\dfrac{\theta\begin{bmatrix}11110\cdots0\\00100\cdots0\end{bmatrix}(u(P),\Pi)}{\theta\begin{bmatrix}00010\cdots0\\00110\cdots0\end{bmatrix}(u(P),\Pi)}$, etc. We note that denominators in (22) and (23) are coincided.

Since for any complex number $\lambda\neq0$, $\lambda-(\lambda-1)=1$ and hence

$$(24)\qquad \sqrt{\sqrt{\lambda_3}\sqrt{\lambda_3}\sqrt{\lambda_3}\sqrt{\lambda_3}}\pm\sqrt{\sqrt{\lambda_3-1}\sqrt{\lambda_3-1}\sqrt{\lambda_3-1}\sqrt{\lambda_3-1}}=\pm1.$$

Putting (22) and (23) in (24), and then rearranging the terms suitably, we finally obtain a period relation of Schottky type on a hyperelliptic Riemann surface S of genus $g\geq4$.

THEOREM 1. *On a hyperelliptic Riemann surface S of genus $g\geq4$ with a canonical homology basis (Γ,Δ) as shown in Figure, a period relation of Schottky typo* $\sqrt{r_1}\pm\sqrt{r_2}\pm\sqrt{r_3}=0$ *holds, where r_k, $k=1,2,3$, are given by* (5).

4. Poincare's approximate period relation.

As in Section 3, we consider a hyperelliptic Riemann surface S of genus $g\geq4$ endowed with a canonical homology basis (Γ,Δ) as shown in Figure, We assume further that the period matrix Π of S and (Γ,Δ) is very close to diagonal form

$$\tilde{\Pi}=\begin{bmatrix}\pi_{11} & & 0\\ & \pi_{22} & \\ 0 & & \ddots\ \pi_{gg}\end{bmatrix}.$$

Since each Riemann theta constant appeared in a period relation of Schottky type in Theorem 1 is an analytic funciton of entires π_{ij}'s, and hence as of the $\frac{g(g-1)}{2}$ off-diagonal entries π_{ij}'s, $1 \leq i < j \leq g$, of Π.

In case Π is very close to $\tilde{\Pi}$, we can expand each theta constant shown in (5) in a Maclaurin series about "the origin" $\pi_{ij}=0$, $1 \leq i < j \leq g$.

In fact, since

$$\text{(25)} \qquad \theta\begin{bmatrix} \varepsilon_1 \cdots \varepsilon_g \\ \varepsilon'_1 \cdots \varepsilon'_g \end{bmatrix}(\tilde{\Pi}) = \prod_{k=1}^{g} \theta\begin{bmatrix} \varepsilon_k \\ \varepsilon'_k \end{bmatrix}(\pi_{kk}) = \prod_{k=1}^{g} \theta\begin{bmatrix} \varepsilon_k \\ \varepsilon'_k \end{bmatrix}_k \text{ and}$$

$$\text{(26)} \qquad \frac{\partial}{\partial \pi_{ij}} \theta\begin{bmatrix} \varepsilon_1 \cdots \varepsilon_g \\ \varepsilon'_1 \cdots \varepsilon'_g \end{bmatrix}(\tilde{\Pi}) = \frac{1}{2\pi i} \prod_{\substack{k=1 \\ k \neq i,j}}^{g} \theta\begin{bmatrix} \varepsilon_k \\ \varepsilon'_k \end{bmatrix}_k \frac{d}{du} \theta\begin{bmatrix} \varepsilon_i \\ \varepsilon'_i \end{bmatrix}(u, \pi_{ii}) \Big|_{u=0} \times \frac{d}{du} \theta\begin{bmatrix} \varepsilon_j \\ \varepsilon'_j \end{bmatrix}(u, \pi_{jj}) \Big|_{u=0},$$

$$\text{(27)} \qquad \theta\begin{bmatrix} \varepsilon_1 \cdots \varepsilon_g \\ \varepsilon'_1 \cdots \varepsilon'_g \end{bmatrix}(\Pi) = \prod_{k=1}^{g} \theta\begin{bmatrix} \varepsilon_k \\ \varepsilon'_k \end{bmatrix}_k + \sum_{1 \leq i < j \leq g} \left(\frac{\partial}{\partial \pi_{ij}} \theta\begin{bmatrix} \varepsilon_1 \cdots \varepsilon_g \\ \varepsilon'_1 \cdots \varepsilon'_g \end{bmatrix}(\tilde{\Pi}) \cdot \pi_{ij} \right) + 0(\varepsilon^4)$$

$$= \prod_{k=1}^{g} \theta\begin{bmatrix} \varepsilon_k \\ \varepsilon'_k \end{bmatrix}_k + \frac{1}{2\pi i} \sum_{1 \leq i < j \leq g} \left(\prod_{\substack{k=1 \\ k \neq i,j}}^{g} \theta\begin{bmatrix} \varepsilon_k \\ \varepsilon'_k \end{bmatrix}_k \theta'\begin{bmatrix} \varepsilon_i \\ \varepsilon'_i \end{bmatrix}_i \theta'\begin{bmatrix} \varepsilon_j \\ \varepsilon'_j \end{bmatrix}_j \pi_{ij} \right) + 0(\varepsilon^4),$$

where $\varepsilon = \max_{1 \leq i < j \leq g} \{\sqrt{|\pi_{ij}|}\}$ and $\theta'\begin{bmatrix} \varepsilon_k \\ \varepsilon'_k \end{bmatrix}_k = \frac{d}{du} \theta\begin{bmatrix} \varepsilon_k \\ \varepsilon'_k \end{bmatrix}(u, \pi_{kk}) \Big|_{u=0}$, $1 \leq k \leq g$.

Now, appealing to (27) and

$$\text{(28)} \qquad \theta'\begin{bmatrix} 1 \\ 1 \end{bmatrix}(o, \tau) = \frac{d}{du} \theta\begin{bmatrix} 1 \\ 1 \end{bmatrix}(u, \tau) \Big|_{u=0} = -\pi\, \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix} \theta\begin{bmatrix} 1 \\ 0 \end{bmatrix} \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix} \neq 0$$

for any τ in $\mathfrak{S}_1$, we can easily check that from (5)

$$\text{(29)} \qquad \theta\begin{bmatrix} 10100 \cdots 0 \\ 10100 \cdots 0 \end{bmatrix} = \frac{\pi}{2i} \prod_{k=1,3} \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_k \theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}_k \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}_k \cdot \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_2 \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_4 \cdot \prod_{k=5}^{g} \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_k \cdot \pi_{13} + 0(\varepsilon^4),$$

$$\theta\begin{bmatrix} 10010 \cdots 0 \\ 11110 \cdots 0 \end{bmatrix} = \frac{\pi}{2i} \prod_{k=1,4} \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_k \theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}_k \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}_k \cdot \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}_2 \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}_3 \cdot \prod_{k=5}^{g} \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_k \cdot \pi_{14} + 0(\varepsilon^4),$$

$$\theta\begin{bmatrix} 01100 \cdots 0 \\ 01110 \cdots 0 \end{bmatrix} = \frac{\pi}{2i} \prod_{k=2,3} \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_k \theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}_k \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}_k \cdot \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_1 \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}_4 \cdot \prod_{k=5}^{g} \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_k \cdot \pi_{23} + 0(\varepsilon^4),$$

$$\theta\begin{bmatrix} 11010 \cdots 0 \\ 01010 \cdots 0 \end{bmatrix} = \frac{\pi}{2i} \prod_{k=2,4} \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_k \theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}_k \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}_k \cdot \theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}_1 \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_3 \prod_{k=5}^{g} \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_k \cdot \pi_{24} + 0(\varepsilon^4),$$

$$\theta\begin{bmatrix} 00010 \cdots 0 \\ 10000 \cdots 0 \end{bmatrix} = \theta\begin{bmatrix} 0 \\ 1 \end{bmatrix}_1 \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_2 \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_3 \theta\begin{bmatrix} 1 \\ 0 \end{bmatrix}_4 \prod_{k=5}^{g} \theta\begin{bmatrix} 0 \\ 0 \end{bmatrix}_k + 0(\varepsilon^4),$$

$$\theta\begin{bmatrix}00100\cdots0\\11010\cdots0\end{bmatrix}=\theta\begin{bmatrix}0\\1\end{bmatrix}_1\theta\begin{bmatrix}0\\1\end{bmatrix}_2\theta\begin{bmatrix}1\\0\end{bmatrix}_3\theta\begin{bmatrix}0\\1\end{bmatrix}_4\prod_{k=5}^{g}\theta\begin{bmatrix}0\\0\end{bmatrix}_k+0(\varepsilon^4),$$

$$\theta\begin{bmatrix}11100\cdots0\\00000\cdots0\end{bmatrix}=\theta\begin{bmatrix}1\\0\end{bmatrix}_1\theta\begin{bmatrix}1\\0\end{bmatrix}_2\theta\begin{bmatrix}1\\0\end{bmatrix}_3\theta\begin{bmatrix}0\\0\end{bmatrix}_4\prod_{k=5}^{g}\theta\begin{bmatrix}0\\0\end{bmatrix}_k+0(\varepsilon^4),$$

$$\theta\begin{bmatrix}01010\cdots0\\00100\cdots0\end{bmatrix}=\theta\begin{bmatrix}0\\0\end{bmatrix}_1\theta\begin{bmatrix}1\\0\end{bmatrix}_2\theta\begin{bmatrix}0\\1\end{bmatrix}_3\theta\begin{bmatrix}1\\0\end{bmatrix}_4\prod_{k=5}^{g}\theta\begin{bmatrix}0\\0\end{bmatrix}_k+0(\varepsilon^4).$$

Note that $\theta\begin{bmatrix}1\\1\end{bmatrix}_k=\theta'\begin{bmatrix}0\\0\end{bmatrix}_k=\theta'\begin{bmatrix}1\\0\end{bmatrix}_k=\theta'\begin{bmatrix}0\\1\end{bmatrix}_k=0$, $1\leq k\leq g$. Multiplying together, we obtain

$$(30)\qquad r_1=\frac{\pi^4}{16}\prod_{k=1}^{4}\left(\theta\begin{bmatrix}0\\0\end{bmatrix}_k\theta\begin{bmatrix}1\\0\end{bmatrix}_k\theta\begin{bmatrix}0\\1\end{bmatrix}_k\right)^4\cdot\prod_{k=5}^{g}\theta^8\begin{bmatrix}0\\0\end{bmatrix}_k\cdot\pi_{13}\pi_{14}\pi_{23}\pi_{24}+0(\varepsilon^{10}).$$

For r_2 and r_3, the similar computations give the expressions

$$(31)\qquad r_2=\frac{\pi^4}{16}\prod_{k=1}^{4}\left(\theta\begin{bmatrix}0\\0\end{bmatrix}_k\theta\begin{bmatrix}1\\0\end{bmatrix}_k\theta\begin{bmatrix}0\\1\end{bmatrix}_k\right)^4\cdot\prod_{k=5}^{g}\theta^8\begin{bmatrix}0\\0\end{bmatrix}_k\cdot\pi_{12}\pi_{14}\pi_{32}\pi_{34}+0(\varepsilon^{10}),$$

$$(32)\qquad r_3=\frac{\pi^4}{16}\prod_{k=1}^{4}\left(\theta\begin{bmatrix}0\\0\end{bmatrix}_k\theta\begin{bmatrix}1\\0\end{bmatrix}_k\theta\begin{bmatrix}0\\1\end{bmatrix}_k\right)^4\cdot\prod_{k=5}^{g}\theta^8\begin{bmatrix}0\\0\end{bmatrix}_k\cdot\pi_{12}\pi_{13}\pi_{42}\pi_{43}+0(\varepsilon^{10}).$$

By Theorem 1, we know that on a hyperelliptic Riemann surface S of genus $g\geq4$

$$(33)\qquad \begin{aligned}\sqrt{r_1}\pm\sqrt{r_2}\pm\sqrt{r_3}&=A\Big(\sqrt{\pi_{13}\pi_{14}\pi_{23}\pi_{24}+0(\varepsilon^{10})}\\&\qquad\pm\sqrt{\pi_{12}\pi_{14}\pi_{32}\pi_{34}+0(\varepsilon^{10})}\\&\qquad\pm\sqrt{\pi_{12}\pi_{13}\pi_{42}\pi_{43}+0(\varepsilon^{10})}\Big)\\&=0\end{aligned}$$

and $A=\frac{\pi^2}{4}\prod_{k=1}^{4}\left(\theta\begin{bmatrix}0\\0\end{bmatrix}_k\theta\begin{bmatrix}1\\0\end{bmatrix}_k\theta\begin{bmatrix}0\\1\end{bmatrix}_k\right)^2\cdot\prod_{k=5}^{g}\theta^4\begin{bmatrix}0\\0\end{bmatrix}_k\neq0$,

and thus we have a relation (6) among π_{ij} in Π.

THEOREM 2. *On a hyperelliptic Riemann surface S of genus $g\geq4$ with a canonical homology basis (Γ, Δ) as shown in Figure such that the period matrix Π of S and (Γ, Δ) is close to diagonal form, a Poincaré's approximate period relation (6) exists.*

We remark that our result of Theorem 2 of course includes the result done by Rauch [3] for genus $g=4$.

Republic of Korea Naval Academy, Korea

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