Kyungpook Math. J.
Volume 13, Number 2
December, 1973

THE FINITE SQUARE SEMI-UNIFORMITY

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1. Introduction.

Let $(X, \mathscr{T})$ be a topological space and define $\mathfrak{S}(\mathscr{T})$ to be $\{S : S=O_1\times O_1\cup O_2\times O_2$ where $O_i\in\mathscr{T}$ and $X=O_1\cup O_2\}$ and let $\mathscr{U}(\mathscr{T})$ be the semi-uniformity for X generated by $\mathfrak{S}(\mathscr{T})$ as subbase.

In this paper an attempt is made to get relationships between $\mathscr{T}$ and $\mathscr{U}(\mathscr{T})$. $\mathscr{T}(\mathscr{U}(\mathscr{T}))$ will denote $\{O^* : x\in O^*$ implies that there exists a $U\in\mathscr{U}(\mathscr{T})$ such that $U[x]\subset O^*\}$.

THEOREM 1.1 *$\mathscr{T}(\mathscr{U}(\mathscr{T}))\subset\mathscr{T}$.*

PROOF. Let $x\in O^*\in\mathscr{T}(\mathscr{U}(\mathscr{T}))$. There exists then a $U\in\mathscr{U}(\mathscr{T})$ such that $U[x]\subset O^*$. But $U\supset S_1\cap\cdots\cap S_n$ where $S_i\in\mathfrak{S}(\mathscr{T})$. Thus $x\in S_1[x]\cap\cdots\cap S_n[x]\subset U[x]\subset O^*$ and each $S_i[x]\in\mathscr{T}$. Thus $O^*\in\mathscr{T}$.

In theorem 3.1, a necessary and sufficient condition is given for $\mathscr{T}=\mathscr{T}(\mathscr{U}(\mathscr{T}))$.

Let $\mathscr{B}(\mathscr{T})=\{B : B=O_1\times O_1\cup\cdots\cup O_n\times O_n$ where $O_i\in\mathscr{T}$ and $X=O_1\cup\cdots\cup O_n\}$.

THEOREM 1.2 *$\mathscr{B}(\mathscr{T})$ is a base for $\mathscr{U}(\mathscr{T})$.*

PROOF. We first show that $\mathscr{B}(\mathscr{T})\subset\mathscr{U}(\mathscr{T})$. Let $X=O_1\cup\cdots\cup O_n$, $O_i\in\mathscr{T}$. For each $\sigma\subset\{1, \cdots, n\}$, let $G\sigma=\cup\{O_i : i\in\sigma\}$. Then $O_1\times O_1\cup\cdots\cup O_n\times O_n\supset\cap\{G\sigma\times G\sigma \cup G_{\mathscr{C}\sigma}\times G_{\mathscr{C}\sigma} : \sigma\subset\{1, \cdots, n\}\}\in\mathscr{U}(\mathscr{T})$. Thus $O_1\times O_1 \cup\cdots\cup O_n\times O_n\in\mathscr{U}(\mathscr{T})$. Let $X=O_i\cup U_i$, $1\le i\le n$, where $O_i\in\mathscr{T}$ and $U_i\in\mathscr{T}$. For each $\sigma\subset\{1, \cdots, n\}$, let $G\sigma=\cap\{O_i : i\in\sigma\}$ and $H\sigma=\cap\{U_i : i\in\sigma\}$. Then $\cap\{O_i\times O_i\cup U_i\times U_i : 1\le i\le n\}\supset \{G\sigma\cap H\sigma\delta\times G\delta\cap H\sigma\delta : \sigma\subset\{1, \cdots, n\}\in\mathscr{B}(\mathscr{T})$. Hence $\mathscr{B}(\mathscr{T})$ has the base property.

COROLLARY 1.3 *$\Delta\in\mathscr{U}(\mathscr{T})$ iff $(X, \mathscr{T})$ is finite and discrete, Δ denoting the diagonal in $X\times X$.*

PROOF. If $(X, \mathscr{T})$ is finite and discrete, then $\Delta=\cup\{\{x\}\times\{x\} : x\in X\}\in\mathscr{B}(\mathscr{T})$.

Conversely, let $\Delta \in \mathscr{U}(\mathscr{T})$. Then $\Delta \supset O_1 \times O_1 \cup \cdots \cup O_n \times O_n$ where $O_i \in \mathscr{T}$ and $X = \bigcup \{O_i : 1 \leq i \leq n\}$. Then each O_i is a singleton or is empty. Thus X is finite and $\mathscr{T}$ is discrete.

THEOREM 1.4 *A topology $\mathscr{T}$ is a chain (linearly ordered by inclusion) iff $\mathscr{U}(Y \cap \mathscr{T}) = \{Y \times Y\}$ for all $Y \subset X$.*

PROOF. Let be $\mathscr{T}$ a chain and suppose hat $Y \subset X$. Suppose further that $Y = (Y \cap O_1) \cup (Y \cap O_2)$ with $O_i \in \mathscr{T}$. We may assume that $O_1 \subset O_2$. It follows then that $(O_1 \cap Y) \times (O_1 \cap Y) \cup (O_2 \cap Y) \times (O_2 \cap Y) = Y \times Y$ and hence $\mathscr{U}(Y \cap \mathscr{T}) = \{Y \times Y\}$.

Conversely, suppose that $\mathscr{U}(Y \cap \mathscr{T}) = \{Y \times Y\}$ for all $Y \subset X$. If $\mathscr{T}$ is not a chain, there exist O_i in $\mathscr{T}$ such that $O_1 \not\subset O_2$ and $O_2 \not\subset O_1$; let $Y = O_1 \cup O_2$. Then $O_i \in Y \cap \mathscr{T}$, but $O_1 \times O_1 \cup O_2 \times O_2 \neq Y \times Y$ and hence $\mathscr{U}(Y \cap \mathscr{T}) \neq \{Y \times Y\}$, a contradiction.

THEOREM 1.5 *$\mathscr{U}(\mathscr{T}) = \{X \times X\}$ iff $X = O_1 \cup O_2$, $O_i \in \mathscr{T}$ implies that $X = O_1$ or $X = O_2$.*

PROOF. Suppose that $\mathscr{U}(\mathscr{T}) = \{X \times X\}$ and that $X = O_1 \cup O_2$, $O_i \in \mathscr{T}$. If $O_i \neq X$ for $i = 1, 2$, then $O_1 \times O_1 \cup O_2 \times O_2 \in \mathscr{U}(\mathscr{T})$, but $O_1 \times O_1 \cup O_2 \times O_2 \neq X \times X$.

The converse is clear.

THEOREM 1.6 *$(X, \mathscr{T})$ is connected iff $X \times X$ is the only equivalence relation in $\mathscr{U}(\mathscr{T})$.*

PROOF. If $(X, \mathscr{T})$ is not connected, there exist O_1, O_2 disjoint, nonempty open sets such that $X = O_1 \cup O_2$. Then $O_1 \times O_1 \cup O_2 \times O_2$ is an equivalence relation in $\mathscr{U}(\mathscr{T})$ which is different from $X \times X$.

Conversely, suppose E is an equivalence relation in $\mathscr{U}(\mathscr{T})$ which is different from $X \times X$. By theorem 1.2, there exist open sets O_i, $1 \leq i \leq n$ such that $X = O_1 \cup \cdots \cup O_n$ and $E \supset O_1 \times O_1 \cup \cdots \cup O_n \times O_n$. Take $x \in X$; let $A = \bigcup \{O_i : O_i \cap E[x] \neq \phi\}$ and let $B = \bigcup \{O_i : O_i \cap E[x] = \phi\}$. Note firstly that if $O_i \cap E[x] \neq \phi$, then $O_i \cap E[x]$. To see this, let $y \in O_i \cap E[x]$. Then $E[x] = E[y] \supset (O_1 \times O_1 \cup \cdots \cup O_n \times O_n)[y] \supset O_i$. It follows then that $\phi \neq A \subset E[x]$ and A is open. Hence $\phi \neq B \in \mathscr{T}$ and $A \cap B = \phi$, $X = A \cup B$. Thus $(X, \mathscr{T})$ is disconnected.

THEOREM 1.7 *Let $(X, \mathscr{T})$ be a topological space and $Y \subset X$. Then* (i) *$Y \times Y \cap \mathscr{U}(\mathscr{T}) \subset \mathscr{U}(Y \cap \mathscr{T})$ and* (ii) *if Y is closed, then equality holds.*

PROOF. (i) Let $U \in \mathscr{U}(\mathscr{T})$: by theorem 1.2, $U \supset O_1 \times O_1 \cup \cdots \cup O_n \times O_n$ where $O_i \in \mathscr{T}$ and $X = O_1 \cup \cdots \cup O_n$. Then $Y \times Y \cap U \supset (Y \cap O_1) \times (Y \cap O_1) \cup \cdots \cup (Y \cap O_n) \times (Y \cap O_n)$ and $Y \times Y \cap U \in \mathscr{U}(Y \cap \mathscr{T})$. (ii) Let Y be closed and suppose $Y = (Y \cap O_1) \cup (Y \cap O_2)$ where $O_i \in \mathscr{T}$. Then $(Y \cap O_1) \times (Y \cap O_1) \cup (Y \cap O_2) \times (Y \cap O_2) \supset Y \times Y \cap (O_1 \times O_1 \cup O_2 \times O_2 \cup \mathscr{C} Y \times \mathscr{C} Y) \in Y \times Y \cap \mathscr{U}(\mathscr{T})$.

2. Separation Properties.

THEOREM 2.1 *$(X, \mathscr{T})$ is a T_1-space iff $\cap \mathscr{U}(\mathscr{T}) = \Delta$.*

PROOF. Suppose that $(X, \mathscr{T})$ is a T_1-space and $x \neq y$. Then $(x, y) \notin \mathscr{C}\{x\} \times \mathscr{C}\{x\} \cup \mathscr{C}\{y\} \times \mathscr{C}\{y\} \in \delta(\mathscr{T}) \subset \mathscr{U}(\mathscr{T})$ and $\Delta = \cap \mathscr{U}(\mathscr{T})$.

Conversely, suppose that $\Delta = \cap \mathscr{U}(\mathscr{T})$. We will show that $\mathscr{C}\{x\} \in \mathscr{T}$ for each x. Let $y \in \mathscr{C}\{x\}$: then $x \neq y$ and hence there exist open sets O_i such that $X = O_1 \cup O_2$ and $(x, y) \notin O_1 \times O_1 \cup O_2 \times O_2$. If $x \in O_1$, then $y \notin O_1$ and $y \in O_2 \subset \mathscr{C}\{x\}$; if $x \notin O_1$, then $x \in O_2$ and $y \notin O_2$ and $y \in O_1 \subset \mathscr{C}\{x\}$.

A space $(X, \mathscr{T})$ is called a $T_{2.5}$-*space* iff $x \neq y$ implies that there exist open sets O_1 and O_2 such that $x \in O_1$, $y \in O_2$ and $\mathrm{c}(O_1) \cap \mathrm{c}(O_2) = \phi$, c denoting the closure operator.

THEOREM 2.2 *A space $(X, \mathscr{T})$ is a $T_{2.5}$-space iff $\Delta = \cap \{\mathrm{c}U : U \in \mathscr{U}(\mathscr{T})\}$.*

PROOF. Suppose that $(X, \mathscr{T})$ is a $T_{2.5}$-space and $x \neq y$. There exist then open sets O_1 and O_2 such that $x \in O_1$, $y \in O_2$ and $\mathrm{c}O_1 \cap \mathrm{c}O_2 = \phi$. Then $X = \mathscr{C}\mathrm{c}O_1 \cup \mathscr{C}\mathrm{c}O_2$, but $(x, y) \notin \mathrm{c}\mathscr{C}\mathrm{c}O_1 \times \mathrm{c}\mathscr{C}\mathrm{c}O_1 \cup \mathrm{c}\mathscr{C}\mathrm{c}O_2 \times \mathrm{c}\mathscr{C}\mathrm{c}O_2$ since $y \notin \mathrm{c}\mathscr{C}\mathrm{c}O_2$ and $x \notin \mathrm{c}\mathscr{C}\mathrm{c}O_1$. Thus $(x, y) \notin \cap \{\mathrm{c}U : U \in \mathscr{U}(\mathscr{T})\}$.

Conversely, suppose that $\Delta = \cap \{\mathrm{c}U : U \in \mathscr{U}(\mathscr{T})\}$ and $x \neq y$. Then $(x, y) \notin \mathrm{c}U$ for some $U \in \mathscr{U}(\mathscr{T})$. Then by theorem 2.1, $U \supset O_1 \times O_1 \cup \cdots \cup O_n \times O_n$ where $O_i \in \mathscr{T}$ and $X = O_1 \cup \cdots \cup O_n$. Hence $(x, y) \in A \times B \subset \mathscr{C}\mathrm{c}U \subset \mathscr{C}(O_1 \times O_1 \cup \cdots \cup O_n \times O_n) \subset \mathscr{C}\Delta$ where A and B are in $\mathscr{T}$. Then $\mathrm{c}A \times \mathrm{c}B \subset \mathscr{C}\Delta$ and hence $\mathrm{c}A \cap \mathrm{c}B = \phi$. It follows then that $(X, \mathscr{T})$ is a $T_{2.5}$-space.

THEOREM 2.3 *A space $(X, \mathscr{T})$ is normal iff $\{\mathrm{c}U : U \in \mathscr{U}(\mathscr{T})\}$ is a base for $\mathscr{U}(\mathscr{T})$.*

PROOF Let $(X, \mathscr{T})$ be normal and suppose that $V \in \mathscr{U}(\mathscr{T})$. Then $V \supset O_1 \times O_1 \cup \cdots \cup O_n \times O_n$ where $O_i \in \mathscr{T}$ and $X = O_1 \cup \cdots \cup O_n$. Since $(X, \mathscr{T})$ is normal, there exist open sets $O_1^* \cdots, O_n^*$ which cover X and $\mathrm{c}O_i^* \subset O_i$. Thus letting $U = O_1^* \times O_1^* \cup \cdots \cup O_n^* \times O_n^*$, it follows that $V \supset \mathrm{c}U$ and $U \in \mathscr{U}(\mathscr{T})$.

Conversely, let $\{cU : U \in \mathscr{U}(\mathscr{T})\}$ be a base for $\mathscr{U}(\mathscr{T})$. To show that $(X, \mathscr{T})$ is normal, let $X = O_1 \cup O_2$ where $O_i \in \mathscr{T}$. It suffices to find closed sets E_1 and E_2 which cover X and for which $E_i \subset O_i$. Now $O_1 \times O_1 \cup O_2 \times O_2 \in \mathscr{U}(\mathscr{T})$ and hence contains $cO_1^* \times cO_1^* \cup \cdots \cup cC_n^* \times cO_n^*$ for some open cover $O_1^*, \cdots, O_n^*$ of X. It is clear that $cO_i^* \times cO_i^* \subset O_1 \times O_1$ or $cO_i^* \times cO_i^* \subset O_2 \times O_2$ for each i. Let $E_1 = \cup \{cO_i^* : cO_i^* \times cO_i^* \subset O_1 \times O_1\}$ and $E_2 = \cup \{cO_i^* : cO_i^* \times cO_i^* \subset O_2 \times O_2\}$. It is clear that E_1 and E_2 are the required sets.

3. R_0-spaces.

A space $(X, \mathscr{T})$ is called an R_0-*space* iff $x \in O \in \mathscr{T}$ implies that $c(x) \subset O$.

THEOREM 3.1 *A space $(X, \mathscr{T})$ is an R_0-space iff $\mathscr{T}(\mathscr{U}(\mathscr{T})) = \mathscr{T}$.*

PROOF. Suppose that $(X, \mathscr{T})$ is an R_0 space. By theorem 1.1, it suffices to show that $\mathscr{T} \subset \mathscr{T}(\mathscr{U}(\mathscr{T}))$: let $x \in O \in \mathscr{T}$. Then $c(x) \subset O$ and $X = O \cup \mathscr{C}c(x)$. Hence $O \times O \cup \mathscr{C}c(x) \times \mathscr{C}c(x) \in \mathscr{U}(\mathscr{T})$ and $(O \times O \cup \mathscr{C}c(x) \times \mathscr{C}c(x))[x] = O$.

Conversely, suppose that $\mathscr{T} = \mathscr{T}(\mathscr{U}(\mathscr{T}))$ and let $x \in O \in \mathscr{T}$. Let $y \in c(x)$: we will show that $y \in O$. There exists a $U \in \mathscr{U}(\mathscr{T})$ such that $U[x] \subset O$. By theorem 1.2, there exists a symmetric open set G which contains the diagonal and is contained in U. Then $G[y]$ is open and hence $G[y] \cap \{x\} \neq \phi$. Thus $y \in G[x] \subset U[x] \subset O$.

THEOREM 3.2 *Suppose that $\mathscr{T}_1$ and $\mathscr{T}_2$ are topologies for X. Then* (1) *if $\mathscr{T}_1 \subset \mathscr{T}_2$, then $\mathscr{U}(\mathscr{T}_1) \subset \mathscr{U}(\mathscr{T}_2)$ and* (2) *if $\mathscr{T}_1$ is R_0 and $\mathscr{U}(\mathscr{T}_1) \subset \mathscr{U}(\mathscr{T}_2)$, then $\mathscr{T}_1 \subset \mathscr{T}_2$.*

PROOF. (1) is clear. (2) Let $x \in O \in \mathscr{T}_1$. Then $c_1(x) \subset O$, c_1 denoting the closure operator relative to $\mathscr{T}_1$. Then $O \times O \cup \mathscr{C}c_1(x) \times \mathscr{C}c_1(x) \in \mathscr{U}(\mathscr{T}_1) \subset \mathscr{U}(\mathscr{T}_2)$. Hence there exists a $G \in \mathscr{T}_2 \times \mathscr{T}_2$ such that $O \times O \cup \mathscr{C}c_1(x) \times \mathscr{C}c_1(x) \supset G \supset \Delta$. Thus $O = (O \times O \cup \mathscr{C}c_1(x) \times \mathscr{C}c_1(x))[x] \supset G[x] \supset \{x\}$. But $G[x] \in \mathscr{T}_2$ and $x \in G[x] \subset O$. It follows then that $O \in \mathscr{T}_2$.

COROLLARY 3.3 *Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be R_0-topologies for X. Then $\mathscr{T}_1 = \mathscr{T}_2$ iff $\mathscr{U}(\mathscr{T}_1) = \mathscr{U}(\mathscr{T}_2)$.*

THEOREM 3.4 *Let $(X, \mathscr{T})$ be an R_0-space. Then $\mathscr{T} = \{\phi, X\}$ iff $\mathscr{U}(\mathscr{T}) = \{X \times X\}$.*

PROOF. If $\mathscr{T} = \{\phi, X\}$, then clearly $\mathscr{U}(\mathscr{T}) = \{X \times X\}$.

Conversely, suppose that $\mathscr{U}(\mathscr{T}) = \{X \times X\}$ and that $\phi \neq O \in \mathscr{T}$; let $x \in X$. It follows then that $X\times X=O\times O\cup\mathscr{C}c(x)\times\mathscr{C}c(x)$ and hence $X=(O\times O\cup\mathscr{C}c(x)\times\mathscr{C}c(x))[x]=0$.

4. Countability.

THEOREM 4.1 *$\mathscr{U}(\mathscr{T})$ has a countable base if $(X, \mathscr{T})$ is compact and second axiom.*

PROOF. Let $\{O_i : i\in P\}$ be a countable base for $\mathscr{T}$ and suppose that $X=O\cup G$, $O\in\mathscr{T}$, $G\in\mathscr{T}$. Then $X=\cup\{O_i : O_i\subset O \text{ or } O_i\subset G\}$ and since $(X, \mathscr{T})$ is compact, there exist O_{i_j}, $1\leq j\leq n$ such that $X=\cup\{O_{i_j} : 1\leq j\leq n\}$ and $O_{i_j}\subset O$ or $O_{i_j}\subset G$. Hence $O\times O\cup G\times G\supset\cup\{O_{i_j}\times O_{i_j} : 1\leq j\leq n\}\supset\Delta$. Thus $\{\cup\{O_i\times O_i : i\in P^*\subset P,\ P^* \text{ finite}, X=\cup\{O_i : i\in P^*\}\}\}$ is a countable base for $\mathscr{U}(\mathscr{T})$.

THEOREM 4.2 *$(X, \mathscr{T})$ is a second axiom space if $\mathscr{U}(\mathscr{T})$ has a countable base and $(X, \mathscr{T})$ is an R_0-space.*

PROOF. Let $\{U_i : i\in P\}$ be a countable base for $\mathscr{U}(\mathscr{T})$. By theorem 1.2, for each integer i, there exist open sets O^i_j, $1\leq j\leq n_i$ such that $U_i\supset\cup\{O^i_j\times O^i_j : 1\leq j\leq n_i\}\supset\Delta$. Then $\{O^i_j : 1\leq j\leq n_i,\ i\in P\}$ is a countable base for $\mathscr{T}$. To see this, let $x\in O\in\mathscr{T}$. Then for some integer i, $O\times O\cup\mathscr{C}c(x)\times\mathscr{C}c(x)\supset U_i\supset\cup\{O^i_j\times O^i_j : 1\leq j\leq n_i\}$. Then $x\in O^i_j$ for some j and $x\in O^i_j\subset O$.

5. $\mathscr{U}(\mathscr{T})$ a uniformity generated by equivalence relations.

LEMMA 5.1 *Suppose that $A_1\times A_1\cup\cdots\cup A_n\times A_n\in\mathscr{U}(\mathscr{T})$, $A_i\cap A_j=\phi$ when $i\neq j$ and that $X=A_1\cup\cdots\cup A_n$. Then each A_i is open and closed.*

PROOF. It suffices to show that each A_i is open. By theorem 1.2, $A_1\times A_1\cup\cdots\cup A_n\times A_n\supset O_1\times O_1\cup\cdots\cup O_m\times O_m\supset\Delta$ where $O_i\in\mathscr{T}$. Let $x\in A_i$. Then $x\in O_j$ for some j. Thus $A_i=(A_1\times A_1\cup\cdots\cup A_n\times A_n)[x]\supset(O_j\times O_j)[x]=O_j\supset\{x\}$.

An equivalence relation E on a set X is termed of *finite character* iff $\{E[x] : x\in X\}$ is finite.

THEOREM 5.2 *$\mathscr{U}(\mathscr{T})$ has a base of equivalence relations of finite character iff for each closed set E and each open set O for which $E\subset O$, there exists a clopen set C such that $E\subset C\subset O$.*

PROOF. Sufficiency. Let $X=O_1\cup O_2$, O_i being open. Then $\mathscr{C}O_2\subset O_1$ and hence $\mathscr{C}O_2\subset C\subset O_1$ for some clopen set C. Hence $O_1\times O_1\cup O_2\times O_2\supset C\times C\cup\mathscr{C}C\times\mathscr{C}C\in\mathscr{U}$

$(\mathscr{T})$. It follows then that each $U \in \mathscr{U}(\mathscr{T})$ contains a finite intersection of sets of the form $C \times C \cup \mathscr{C}C \times \mathscr{C}C$. Such finite intersections are equivalence relations of finite character.

Necessity. Let $E \subset O$, E being closed and O being open. Then $O \times O \cup \mathscr{C}E \times \mathscr{C}E \in \mathscr{U}(\mathscr{T})$ and hence $O \times O \cup \mathscr{C}E \times \mathscr{C}E \supset A_1 \times A_1 \cup \cdots \cup A_n \times A_n$ where $A_i \cap A_j = \phi$ when $i \neq j$ and $A_1 \times A_1 \cup \cdots \cup A_n \times A_n \in \mathscr{U}(\mathscr{T})$. By lemma 5.1, each A_i is clopen. Let $O^* = \cup \{A_i : A_i \subset O\}$. O^* is clearly clopen; it suffices to show that $E \subset O^*$. Let $x \in E$: then $x \in A_i$ for some i. It suffices to show that $A_i \subset O$. Suppose $A_i \not\subset O$: take $a \in A_i - O$. Then $(x, a) \in A_i \times A_i \subset O \times O \cup \mathscr{C}E \times \mathscr{C}E$. But $(x, a) \notin O \times O \cup \mathscr{C}E \times \mathscr{C}E$, a contradiction.

COROLLARY 5.3 *Let $(X, \mathscr{T})$ be compact and zero dimensional. Then $\mathscr{U}(\mathscr{T})$ has a base of equivalence relations of finite character.*

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