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ON THE GEOMETRIC MEANS OF AN ENTIRE FUNCTION OF TWO COMPLEX VARIABLES*

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1. Let

$$f(z_1,\ z_2)=\sum_{m_1,m_2\geq 0} a_{m_1,m_2} z_1^{m_1} z_2^{m_2}$$

be an entire function of two complex variables z_1 and z_2, holomorphic for $|z_t|\leq r_t$, $t=1,2$. Let

$$M(r_1,r_2)=\max_{|z_t|\leq r_t} |f(z_1,z_2)|\ ,t=1,2.$$

The Geometric means $G(r_1,\ r_2)$ and $g_k(r_1,\ r_2)$ of the function $|f(z_1,\ z_2)|$ for $|z_t|\leq r_t$ $(t=1,2)$ have been defined as ([2])

$$G(r_1,r_2)=\exp\left\{\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\log|f(r_1e^{i\theta_1},r_2e^{i\theta_2})|d\theta_1\ d\theta_2\right\} \tag{1.1}$$

and

$$g_k(r_1,r_2)=\exp\left\{\frac{(k+1)^2}{(r_1r_2)^{k+1}}\int_0^{r_1}\int_0^{r_2}(x_1x_2)^k\log\ G(x_1,x_2)\ dx_1\ dx_2\right\},$$

where $0<k<\infty$.

In this paper we have obtained a few properties of the Geometric means $G(r_1,r_2)$ and $g_k(r_1,r_2)$.

2. THEOREM 1. *If $f(z_1,z_2)$ is an entire function, other than a constant, then*

$$\lim_{r_1,r_2\to\infty}\left\{\frac{1}{\log\ g_k(r_1,r_2)-(a_1a_2)^{k+1}\log\ g_k(a_1r_1,a_2r_2)}\right\}=0, \tag{2.1}$$

where a_1, a_2 $(0<a_1,a_2<1)$ are constants.

In order to prove this theorem, we need the following result.

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LEMMA 1. *Let $f(z_1, z_2)$ be an entire function, then for* $0<r_1'<R_1'<R_1$ *and* $0<r_2'<R_2'<R_2$,

$$\log G(R_1', r_2')\{(R_1')^{k+1}-(R_1')^{k+1}\}\{(R_2')^{k+1}-(r_2')^{k+1}\}+\{(R_2)^{k+1}-(R_2')^{k+1}\}$$
$$\log G(r_1', R_2')\{(R_1')^{k+1}-(r_1')^{k+1}\}+\log G(R_1', R_2')$$
$$\{(R_1)^{k+1}-(R_1')^{k+1}\}\{(R_2)^{k+1}-(R_2')^{k+1}\}$$

$$\leq (R_1R_2)^{k+1}\log g_k(R_1, R_2)-(R_1'R_2')^{k+1}\log g_k(R_1', R_2') \tag{2.2}$$
$$\leq \log G(R_1, R_2')\,(R_2')^{k+1}\,\{(R_1)^{k+1}-(R_1')^{k+1}\}+$$
$$\log G(R_1', R_2)\,(R_1')^{k+1}\,\{(R_2)^{k+1}-(R_2')^{k+1}\}+$$
$$\log G(R_1, R_2)\,\{(R_2)^{k+1}-(R_2')^{k+1}\}\,\{(R_1)^{k+1}-(R_1')^{k+1}\},$$

where $0<k<\infty$.

PROOF. From (1.1) and (1.2), we have

$$\frac{(r_1r_2)^{k+1}}{(k+1)^2}\log g_k(r_1, r_2)=\int_0^{r_1}\int_0^{r_2}(x_1x_2)^k\log G(x_1, x_2)dx_1\,dx_2. \tag{2.3}$$

The function $G(r_1, r_2)$ is an increasing function of (i) r_1 for a given r_2, (ii) r_2 for a given r_1, and (iii) r_1 and r_2 both increasing.

Now for $0<r_1'<R_1'<R_1$ and $0<r_2'<R_2'<R_2$,

$$(R_1R_2)^{k+1}\log g_k(R_1, R_2)-(R_1'R_2')^{k+1}\log g_k(R_1', R_2')$$
$$=(k+1)^2\left\{\int_{R_1'}^{R_1}\int_0^{R_2'}+\int_0^{R_1'}\int_{R_2'}^{R_2}+\int_{R_1'}^{R_1}\int_{R_2'}^{R_2}\right\}(x_1x_2)^k\log G(x_1, x_2)\,dx_1\,dx_2$$
$$\leq (R_2')^{k+1}\{(R_1)^{k+1}-(R_1')^{k+1}\}\log G(R_1, R_2')+ \tag{2.4}$$
$$(R_1')^{k+1}\{(R_2)^{k+1}-(R_2')^{k+1}\}\log G(R_1', R_2)+$$
$$\{(R_2)^{k+1}-(R_2')^{k+1}\}\,\{(R_1)^{k+1}-(R_1')^{k+1}\}\log G(R_1, R_2).$$

Also

$$(R_1R_2)^{k+1}\log g_k(R_1, R_2)-(R_1'R_2')^{k+1}\log g_k(R_1', R_2')$$
$$\geq (k+1)^2\left\{\int_{R_1'}^{R_1}\int_{r_2'}^{R_2'}+\int_{r_1'}^{R_1'}\int_{R_2'}^{R_2}+\int_{R_1'}^{R_1}\int_{R_2'}^{R_2}\right\}(x_1x_2)^k\log G(x_1, x_2)\,dx_1\,dx_2$$
$$\geq \{(R_1)^{k+1}-(R_1')^{k+1}\}\log G(R_1', r_2')\,\{(R_2')^{k+1}-(r_2')^{k+1}\}+ \tag{2.5}$$
$$\log G(r_1', R_2')\,\{(R_1')^{k+1}-(r_1')^{k+1}\}\,\{(R_2)^{k+1}-(R_2')^{k+1}\}+$$
$$\{(R_1)^{k+1}-(R_1')^{k+1}\}\,\{(R_2)^{k+1}-(R_2')^{k+1}\}\log G(R_1', R_2').$$

Combining (2.4) and (2.5), the result follows.

PROOF OF THEOREM 1. Let us set $R_1=r_1$, $R_1'=a_1r_1$, $r_1'=b_1r_1$, and $R_2=r_2$, $R_2'=a_2r_2$, $r_2'=b_2r_2$ in Lemma 1. Then for $0<a_1, a_2, b_1, b_2<1$,

$$\begin{aligned}
&\log G(a_1r_1, b_2r_2)\ \{1-(a_1)^{k+1}\}\ \{(a_2)^{k+1}-(b_2)^{k+1}\}+\\
&\log G(b_1r_1, a_2r_2)\ \{(a_1)^{k+1}-(b_1)^{k+1}\}\ \{1-(a_2)^{k+1}\}+\\
&\log G(a_1r_1, a_2r_2)\ \{1-(a_1)^{k+1}\}\ \{1-(a_2)^{k+1}\}\\
&\leq \log g_k(r_1, r_2)-(a_1a_2)^{k+1}\log g_k(a_1r_1, a_2r_2)\\
&\leq \log G(r_1, a_2r_2)\ (a_2)^{k+1}\ \{1-(a_1)^{k+1}\}+\log G(a_1r_1, r_2)\ (a_1)^{k+1}\\
&\quad \{1-(a_2)^{k+1}\}+\log G(r_1, r_2)\ \{1-(a_1)^{k+1}\}\ \{1-(a_2)^{k+1}\}.
\end{aligned}$$

Taking the limit as $r_1, r_2\to\infty$, the result follows.

3. THEOREM 2. *Let $f(z_1, z_2)$ be an entire function, then*

$$(3.1)\qquad \limsup_{r_1,r_2\to\infty}\frac{\log g_k(r_1, r_2)}{\log M(r_1, r_2)}\leq \limsup_{r_1,r_2\to\infty}\frac{\log g_k(r_1, r_2)}{\log G(r_1, r_2)}\leq 1.$$

PROOF. From (2.3), we have

$$\begin{aligned}
\log g_k(r_1, r_2)&=\frac{(k+1)^2}{(r_1r_2)^{k+1}}\int_0^{r_1}\int_0^{r_2}(x_1x_2)^k\log G(x_1, x_2)\,dx_1\,dx_2\\
&\leq \log G(r_1, r_2),
\end{aligned}$$

as $G(r_1, r_2)$ is an increasing function of both r_1 and r_2.
Taking limits, we have

$$(3.2)\qquad \limsup_{r_1,r_2\to\infty}\frac{\log g_k(r_1, r_2)}{\log G(r_1, r_2)}\leq 1.$$

Also from (1.1), we get

$$(3.3)\qquad \log G(r_1, r_2)\leq \log M(r_1, r_2).$$

Combining (3.2) and (3.3) the result follows.

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