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SEMIRINGS WITH NONCOMMUTATIVE ADDITION[1)]

By H. E. Stone

With the help of several assumptions weaker than commutativity, factor semirings and cancellative representations of semirings with noncommutative addition are studied, with special attention to lattice preserving properties of homomorphisms. A commutativity theorem is obtained which shows that a large class of semirings are semi-isomorphic to semirings with commutative addition. It is shown that the weakness of semi-isomorphisms as tools in structural investigations may be somewhat remedied by considering semirings which are intrinsically preordered, and a large class of semirings is exhibited for which the structure problem reduces to the study of rings and partially ordered rings, together with a study of semiring extensions.

A triple $(S,+,\cdot)$ is a *right semiring* if $(S,\cdot)$ is a semigroup, $(S,+)$ is a semigroup with identity, and $a(b+c)=ab+ac$ for each a, b, c in S. Left semirings are defined dually. The right and left semirings together form the *near-semirings*, while the *semirings* are both right and left semirings. If $(S,+)$ is a group, we speak of *right* or *left rings, near rings*, and *skewrings*. A skewring with commutative addition is a *ring*.

The operations are extended in the usual way to subsets. A *subsemiring* of a semiring S is a subset T of S such that $0\in T$, $T+T\subset T$, and $T\cdot T\subset T$.

A subsemiring T is an *right ideal* of S provided $ST\subset T$. *Left ideal* is defined dually, and an *ideal* is both a right and left ideal.

A semiring which can be embedded into a ring is called a *halfring*. A most striking and suggestive class of nontrivial halfrings is the collection of positive cones of partially ordered rings, and it will later be seen that the study of halfrings reduces mainly to the study of this class. A general semiring can be regarded as a broad generalization of such a positive cone, and it will be

1) This material is extracted from the author's doctoral dissertation [8] at Texas Christian University. The author is grateful for the assistance and encouragement of his advisor, Professor Ben T. Goldbeck, Jr.

revealing to see to what extent the properties are similar. It will be convenient for efficiency of exposition to consider semirings which actually have a preorder defined on them.

A *preorder* is a reflexive and transitive relation. If a preorder $\leq$ is compatible with the operations of S in the sense that $a\leq b$ implies $a+x\leq b+x$, $x+a\leq x+b$ for all a, b, x in S, and $ax\leq bx$, $xa\leq xb$ for all $x\geq 0$ in S, then S is a *preordered semiring*. For $A\subset S$, let $U(A)=\{s\in S: a\leq s\ \forall a\in A\}$. In particular, $U(0)$ is the *positive cone* of S. If $S=U(0)$, it is *positively ordered*; if $x\leq y$ implies $a+x=y=x+b$ for some a, b in $U(0)$, S is *intrinsically preordered*; if both, S is *naturally preordered*. Since intrinsic preorders most strongly influence the arithmetic of the semiring, it is in them that we will be mostly interested.

A subset T of a semiring S is *centric* in S if $x+S=S+x$ for each x in T. Clearly 0 is always centric in S, but generally S is not centric in itself. It is easy to prove.

PROPOSITION 1. *If P is centric subsemiring of S, the relation $\leq(\mathrm{mod}\ P)$ defined by $x\leq y$ if and only if there exist p, q in S with $x+p=y=q+x$ is an intrinsic preorder on S with positive cone P. Conversely, if $\leq$ is an intrinsic preorder on S, $U(0)$ is a centric subsemiring of S and $\leq$ coincides with $\leq(\mathrm{mod}\ U(0))$.*

A semiring homomorphism η is *monotone* if $x\leq y$ implies $x\eta\leq y\eta$; it is *strictly monotone* (*isotone*) if $x<y$ implies $x\eta<y\eta$ (where $x<y$ means $x\leq y$ and $x\neq y$). If η is monotone, clearly $U(0)\eta\subset U(0\eta)$; this condition is sufficient for η to be monotone if $\leq$ is intrinsic. It is possible that the containment be proper even if η is onto; following [2] we call η an *epimorphism* if it is monotone and $U(0)\eta=U(0\eta)$. An *order-isomorphism* is a monotone homomorphism with monotone inverse; clearly every order-isomorphism is an epimorphism.

Let A and B be subsemirings of S. A is *B-stable* in S provided that for each x, y in S and each b in B, $x+b+y\in A$ implies $x+B+y\subset A$. If A is B-stable, then $0+0+0\in A$ implies $B=0+B+0\subset A$; on the other hand, every subsemiring A is 0-stable. A is called *stable* when it is A-stable.

The *kernel* of a semiring homomorphism is the inverse image of 0. It is easily seen that the kernel of any homomorphism is a stable ideal. Conversely, if I is any ideal, we construct a congruence as in [5] by putting $x\equiv y$ (mod I) if there exists a finite sequence $a_0, \cdots, a_n$ in S with $a_0=x$, $a_n=y$, and for $0\leq i<n$ there exist r_i, s_i in S and p_i, q_i in I such that $r_i+p_i+s_i=a_i$, $r_i+q_i+s_i=a_{i+1}$. I is contained in the congruence class K of 0, and $I=K$ if and only if I is stable.

The set of congruence classes is a factor semiring of S, and will be denoted S/K. The natural map υ carrying x in S onto its congruence class in S/K is a homomorphism. If S is intrinsically preordered with positive cone P, then $P\upsilon$ is centric in S/K, and υ is an epimorphism with respect to the intrinsic preorder induced in S/K by $P\upsilon$.

Following [1], a homomorphism with kernel 0 will be called a *semi-isomorphism*. While a semi-isomorphism is in general not an isomorphism, there has been some interest in their study. If the semiring is intrinsically preordered, every monotone semi-isomorphism is strictly monotone. A standard argument will now prove the following "natural homomorphism theorem."

THEOREM 1. *Let S be an intrinsically preordered semiring, and let η be a monotone homomorphism of S with kernel K. Then the map $\sigma;S/K\rightarrow S\eta$ defined by $(s\upsilon)\sigma=s\eta$ for every s in S is a strictly monotone semi-isomorphism such that $\upsilon\sigma=\eta$, and σ is an epimorphism if and only if η is.*

Let A and B be subsemirings of S, and let s,t belong to S. The *right (A, B)-difference* ${}_A(s\rightarrow t)_B$ is the set $\{x \text{ in } S:\exists\ a\in A,\ b\in B \text{ such that } a+t+x+b=a+s+b\}$. A subsemiring T is *right (A, B)-subtractive* if ${}_A(s\rightarrow t)_B\subset T$ for each s,t in T. *Left (A, B)-difference* is defined dually, and T is *(A, B)-subtractive* if it is both right and left (A, B)-subtractive. If $A=B=0$, the prefix is omitted, and we speak of "difference" and "subtractive" subsemiring; while if $A=B=S$, we speak of "hyperdifference" and "hypersubtractive" subsemirings. If A consists of left and B of right cancellative elements, clearly T is (A, B)-subtractive if and only if it is subtractive. Hypersubtractive ideals generalize the h-ideals of Iizuka [3].

Let T be a stable subsemiring, and let s,t belong to T. If $x+t=s$ then $x+t+0=s$ implies $x+T+0\subset T$, so that $x\in T$. Hence T is subtractive. However, this condition is not in general sufficient for a subsemiring to be stable. A subsemiring T is *normalized* by a subset C of S provided that $\forall c\in C$ and $\forall s\in S$, $c+s\in T$ if and only if $s+c\in T$; this generalizes the concept of normalization in groups. T is *normal* in S if S normalizes T. If T is normal and subtractive in S, and if t, $x+t+y\in T$, then $t+(y+x)\in T$ by normality and $y+x\in T$ by subtractivity. Thus for each $s\in T$, $s+y+x\in T$ and $x+ s+y\in T$ by normality. Hence $x+T+y\subset T$ and T is stable.

A subsemiring T is *C-starlike* for $C\subset S$ if $x+y\in T$ implies $x\in T$ and $y\in T$ for each $x,y\in C$; T is *starlike* if it is *S-starlike*. Clearly every starlike subsemiring is normal and subtractive. If C normalizes T, let $St_C(T)=\{x\in C : \exists y\in C$ with $x+y\in T\}$; this *C-star of T* is a C-starlike subsemiring of C when C is a subsemiring. T is C-starlike if and only if $St_C(T)\subset T$. If S is intrinsically preordered with positive cone P, then P normalizes every subsemiring T, and it is easily seen that T is convex if and only if it is P-starlike. If there is no ambiguity, we write $St_S(T)=T^*$; evidently T^* is a normal starlike subsemiring containing T, and T is starlike if and only if $T^*=T$. If T is a right or left ideal, so is T^*. If T is a normal stable ideal, T^*/T is a skewring.

The observation that each of the families below is closed under arbitrary intersections is sufficient to prove

PROPOSITION 2. *For a semiring S with fixed subsemirings A and B, and fixed subset C, let $\mathscr{L}$ be any of the families of all*

(*a*) *subsemirings of S,*

(*b*) [*right, left*] *(A, B)-subtractive subsemirings of S,*

(*c*) [*A*-] *stable subsemirings of S,*

(*d*) *C-normal subsemirings of S,*

(*e*) *C-starlike subsemirings of S,*

(*f*) [*right, left*] *ideals of S.*

Then $\mathscr{L}$ is a complete lattice with respect to set inclusion.

The following "natural lattice theorem" gives invariance properties of these lattices with respect to the natural homomorphism.

THEOREM 2. *Let S be a semiring, C a subset of S, η a homomorphism of S with kernel K. Let $\mathscr{L}$, respectively $\mathscr{L}'$, be one of the classes of subsemirings of S, respectively $S\eta$, described below, and let $\mathscr{L}_K=\{L\in\mathscr{L} ; K\subset L\}$. Then inverse images under η of elements of $\mathscr{L}'$ belong to $\mathscr{L}_K$. Moreover, if η is the natural homomorphism, then η induces a lattice isomophism of $\mathscr{L}_K$ onto $\mathscr{L}'$.*

(*a*) *$\mathscr{L}$ is all K-stable subsemirings, $\mathscr{L}'$ is all subsemirings,*

(*b*) *$\mathscr{L}$ is all K-stable* [*right, left*] *ideals, $\mathscr{L}'$ is all* [*right, left*] *ideals,*

(*c*) *$\mathscr{L}$ is all stable subsemirings, $\mathscr{L}'$ is all stable subsemirings.*

(*d*) *$\mathscr{L}$ is all K-stable C-normal subsemirings, $\mathscr{L}'$ is all $C\eta$-normal subsemirings,*

(*e*) *$\mathscr{L}$ is all K-stable C-starlike subsemirings, $\mathscr{L}'$ is all $C\eta$-starlike subsemirings.*

PROOF. Let $A\in\mathscr{L}'$, and let $B=\eta^{-1}(A)$. Suppose $s+k+t\in B$ for $s, t\in S$, $k\in K$. Then $s\eta+t\eta\in A$, so that $(s+K+t)\eta\subset A$, whence $s+K+t\subset B$ and B is K-stable. If A is stable and $s+a+t\in B$ for some $a\in B$, then $s\eta+a\eta+t\eta\in A$, whence $s\eta+A+t\eta\subset A$ and $s+B+t\subset B$, so that B is stable. If A is $C\eta$-normal and $x, y\in C$, then $x+y\in B$ implies $x\eta+y\eta\in A$, so that $(y+x)\eta\in A$ and $y+x\in B$. If A is $C\eta$-starlike and x, $y\in C$, then $x+y\in B$ implies $x\eta+y\eta\in A$, so that $x\eta$, $y\eta\in A$; hence x, $y\in B$ and B is C-starlike. The remaining properties are easily seen, so that $B\in\mathscr{L}_K$.

Now let η be the natural homomorphism, and let $A\in\mathscr{L}_K$ be K-stable. Clearly $A\eta$ is a subsemiring of $S\eta$; let $B=\eta^{-1}(A\eta)$. If $b\in B$, then $b\eta=a\eta$ for some $a\in A$. Thus there exist r_i, $s_i\in S$, h_i, $k_i\in K$ such that $r_1+h_1+s_1=b$, $r_n+k_n+s_n=a$, and $r_i+k_i+s_i=r_{i+1}+h_{i+1}+s_{i+1}$ *for* $1\leq i<n$. Since A is K-stable, $r_n+K+s_n\subset A$, and in particular, $r_{n-1}+k_{n-1}+s_{n-1}=r_n+h_n+s_n\in A$. After n steps, $b=r_1+h_1+s_1\in A$, so that $B\subset A$. Since clearly $A\subset B$, η is one-to-one of $\mathscr{L}_K$ onto $\mathscr{L}'$. It is easy to verify that η preserves arbitrary intersections, and so is a lattice isomorphism. The result in the remaining cases follows when it is shown that η preserves those lattices, which is easily done.

We now conclude the study of natural factors of semirings with some isomorphism theorems.

THEOREM 3. *Let K be a stable ideal of a semiring S, and let H be a stable ideal of S/K. If υ is the natural homomorphism of S onto S/K, then $S/\upsilon^{-1}(H)$ is isomorphic to $(S/K)/H$.*

PROOF. Let η be the natural homomorphism of M/K onto $(M/K)/H$, let η' be the natural homomorphism of S onto $S/\upsilon^{-1}(H)$, and define $\phi=\{(x\upsilon\eta,\ x\eta'):x\in S\}$. If $x\upsilon\eta=y\upsilon\eta$, then $x\upsilon\equiv y\upsilon$ (mod H); this is a finite chain of relations $s_{i+1}\upsilon+h_{i+1}\upsilon+t_{i+1}\upsilon=s_i\upsilon+k_i\upsilon+t_i\upsilon$, with $h_i\eta, k_i\eta\in H$. Then h_i, $k_i\in\upsilon^{-1}(H)$, and $(s_{i+1}+h_{i+1}+t_{i+1})\upsilon=(s_i+k_i+t_i)\upsilon$, so that $s_{i+1}+h_{i+1}+t_{i+1}\equiv s_i+k_i+t_i$(mod K). Since $K\subset\upsilon^{-1}(H)$, $s_{i+1}+h_{i+1}+t_{i+1}\equiv s_i+k_i+t_i$ (mod $\upsilon^{-1}(H)$), and $x\eta'=y\eta'$. Thus ϕ is single-valued. As similar argument shows that ϕ is one-to-one. It is clear that ϕ is a homomorphism onto.

COROLLARY. *Let H and K be stable ideals of semiring S, with $K\subset H$. Then S/H is isomorphic to $(S/K)/(H/K)$.*

It is clear in both of these cases that if S is a preordered semiring, both isomorphisms are order-isomorphisms.

In the case of commutative addition, this corollary is LaTorre's Theorem 3.12 in [4]. That theorem has the unnecessary hypothesis that the semiring be "of type (K)."

We now wish to study representations of semirings in cancellative semirings. Throughout, the word "cancellative" will always refer to the additive structure. Recalling the standard definition from semigroup theory, if A is a subsemiring of S, we say A is *left reversible* in S if $S+a \cap A+s \neq \phi$ for each $a \in A$, $s \in S$. *Right reversible* is defined dually, and A is *reversible* if it is both right and left reversible. If A is a left and B a right reversible subsemiring of S, we define a relation $x \cong y$ in S by $x \cong y$ if there exist $a \in A$, $b \in B$ such that $a+x+b=a+y+b$. Using the reversibility, it is easy to see that this is a congruence on S. We define $k_{A,B}$ to be the canonical map of S onto the factor semiring which carries $x \in S$ onto the congruence class of x. The map $k_{A,B}$ is called the (A, B)-*representation* of S. It is easy to prove

THEOREM 4. *Let A be a left and B a right reversible subsemiring of S, and let $k_{A,B}$ be the (A, B)-representation of S. Then $k_{A,B}$ is a homomorphism such that $Ak_{A,B}$ and $Bk_{A,B}$ are left and right cancellative, respectively, in $Sk_{A,B}$. Moreover, any homomorphism having this property factors through $k_{A,B}$.*

For subsets A, B of S, we define the *zeroid of A and B in S* to be $Z(A, B)=\{x \in S : a+x+b=a+b \text{ for some } a \in A,\ b \in B\}$. If A is a left and B a right reversible subsemiring of S, then clearly $Z(A, B)$ is the kernel of $k_{A,B}$, and hence is a stable ideal.

It is easily seen that in this case $Z(A, B)$ is the unique minimum (A, B)-subtractive stable ideal. We will usually write $Z(A)$ for $Z(A, A)$.

If T is a subsemiring of S, and $Z(S) \cap T=0$, then S is called T-precancellative; if S is S-precancellative, it is precancellative. Note that 0 is the unique additive idempotent in a precancellative semiring; it follows easily that 0 is a normal ideal. If S is reversible, it is easily seen that $S/Z(S)$ is precancellative; thus a reversible semiring is precancellative if and only if it is semi-isomorphic to a cancellative semiring.

The lattice preserving properties of $k_{A,B}$ are given in the following "representation lattice theorem."

THEOREM 5. *Let A be a left and B a right reversible subsemiring of S, $C \subset S$, and let K be the (A, B)-representation. Then K induces a lattice isomorphism of the lattice of (A, B)-subtractive subsemirings of S onto the lattice of subtractive subsemirings of Sk. Moreover, the intersections of this lattice with the lattices of stable subsemirings, [right, left] ideals, C-normal and C-starlike subsemirings* *are preserved by the lattice isomorphism.*

PROOF. Note that an (A, B)-subtractive subsemiring T necessarily contains $Z(A, B)$. Let $s, t \in T$, and suppose $xk+sk=tk$. Then $a+(x+s)+b=a+t+b$ for $a \in A$, $b \in B$, and $x \in T$. Thus $xk \in Tk$, and Tk is subtractive. If T' is subtractive in Sk, let $a+x+t+b=a+s+b$ for $a \in A, b \in B$, s, $t \in k^{-1}(T')$. Then $ak+xk+tk+bk=ak+sk+bk$, and cancelling ak and bk, $xk \in T'$ by subtractivity. Hence $x \in k^{-1}(T')$. Now if $x \in k^{-1}(Tk)$, then $xk=tk$ for $t \in T$. Thus $a+x+b=a+t+b$ for $a \in A$, $b \in B$, and $x \in T$. Therefore k is one-to-one as a lattice mapping. The remaining assertions are equally easy to verify.

It is easy to see that if A is right or left reversible in S, then $A\eta$ is right or left reversible in $S\eta$, for any homomorphism η. The natural factor and the (A, B)-representation can now be related by the following "permutation theorem".

THEOREM 6. *Let A be a left and B a right reversible subsemiring of S, and let K be a stable (A, B)-subtractive ideal. Let υ be the natural homomorphism of S onto S/K and let k be the (A, B)-representation. Let υ' be the natural homomorphism of Sk onto Sk/Kk, and let k' be the $(A\upsilon, B\upsilon)$-representation of S/K. Then $S\upsilon k'$ and $Sk\upsilon'$ are isomorphic.*

PROOF. Let $\phi=\{(x\upsilon k', xk\upsilon') : x \in S\}$. *If* $x\upsilon k'=y\upsilon k'$, *then* $(a+x+b)\upsilon=(a+y+b)\upsilon$ for $a \in A$, $b \in B$. Thus there are $h_i, k_i \in K$, r_i, $s_i \in S$ such that $r_i+k_i+s_i=r_{i+1}+h_{i+1}+s_{i+1}$ for $0 \leq i < n$, $a+x+b=r_o+h_o+s_o$, $a+y+b=r_n+k_n+s_n$. Thus $r_ik+k_ik+s_ik=r_{i+1}k+h_{i+1}k+s_{i+1}k$, and since k_ik, $h_ik \in Kk$ for $0 \leq i \leq n$, $(a+x+b)k \equiv (a+y+b)k$ (mod Kk). Thus $xk\upsilon'=yk\upsilon'$, and ϕ is single-valued. It is easily seen that ϕ is a homomorphism of $S\upsilon k'$ onto $Sk\upsilon'$.

Now suppose $xk\upsilon'=yk\upsilon'$. Then there are h_i, $k_i \in K$, r_i, $s_i \in S$ such that $xk=r_ok+h_ok+B_ok$, $yk=r_nk+k_nk+s_nk$, and $(r_i+k_i+s_i)k=(r_{i+1}+h_{i+1}+s_{i+1})k$ for $0 \leq i < n$. Therefore there are $a_i \in A$ and $b_i \in B$ such that $a_i+r_i+k_i+s_i+b_i=a_i+r_{i+1}+h_{i+1}+s_{i+1}+b_i$. By Lemma 1 of [7], there exist $u_o, \cdots, u_n$, $v_o, \cdots, v_n \in S$ such that $u_o+a_o=\cdots=u_n+a_n=c \in A$ and $b_o+v_o=\cdots=b_n+v_n=d \in B$. Hence $c+x+d=u_o+a_o+r_o+h_o+s_o+b_o+v_o \equiv \cdots \equiv u_n+a_n+r_n+k_n+s_n+b_n+v_n=c+y+d$ (mod K), and since $c\upsilon \in A\upsilon$ and $d\upsilon \in B\upsilon$, $x\upsilon k'=y\upsilon k'$. Therefore ϕ is one-to-one.

If $A=B$ is right reversible and cancellative in S. then S can be embedded into a unique monoid $S \to A$ of right differences in which every element of A has an additive inverse and every element of $S \to A$ has form $s-a$ for $s \in S$, $a \in A$: It is easy to see that if S is a semiring, the semiring structure extends to $S \to A$. If A

is both right and left reversible, then $S\to A$ is also a left difference semiring $S\leftarrow A$, and is denoted by $S-A$. In the case $A=B$ is both left and right reversible, then the (A, A)-representation k can be regarded as a mapping *into* $Sk-Ak$; this will be called the A-representation of S. In particular, if S is reversible, it can be represented in a skewring.

It is also well known that if η is a homomorphism of S such that the right reversible cancellative subsemiring A has cancellative image $A\eta$, then there is a unique homomorphism $\bar{\eta}$ of $S\to A$ onto $S\eta\to A\eta$ which extends η. This has an important corollary which we state as

THEOREM 7. *Let A be a reversible cancellative subsemiring of S. Then the stable ideals of S are exactly the intersections with S of the stable ideals of $S-A$.*

Cancellativity has a strong influence on the arithmetic of the semiring. If A is a cancellative left ideal of S, then AS is Abelian. For let ax, $by \in AS$, where $a, b\in A$ and $x, y\in S$. Then $ay+ax+by+bx=(a+b)(x+y)=ay+by+ax+bx$, and since $ay, bx\in A$, they can be cancelled. Hence $ax+by=by+ax$.

If 0 is a normal ideal of S, call S a *normal* semiring. Then the skewring 0^* is a stable ideal of S. Let S' denote the commutator subgroup of 0^*; we shall see in a moment that this definition is natural. We obtain the following striking result.

THEOREM 8. *Let S be a normal semiring. Then $S^1S=SS^1=0$.*

PROOF. Let x, $y\in 0^*$, $z\in S$. Then $(x+y-x-y)z=xz+yz-xz-yz=0$, since 0^* is cancellative and 0^*S is Abelian. Thus $S^1S=0$. Similarly $SS^1=0$.

If S contains even one multiplicatively cancellative element, then all the commutators must be zero, and 0^* is a ring. This result is very old; it just extends the observation that a commutative law of addition is redundant in the definition of a ring with identity. We now generalize it in a way which casts grave doubt on the utility of semi-isomorphisms as tools for studying semiring structure.

Let x be equivalent to y iff there exist $n>0$ and $a_i\in S$, $1\leq i\leq n$, such that $x=\sum a_i$ and $y=\sum a_{i\sigma}$, where σ is some permutation on $\{1, \cdots n\}$. Clearly this is a congruence; let α be the canonical homomorphism of S onto the factor semiring thus determined. This mapping is called the *Abelianizer* of S, and it is easy to see that any homomorphism of S onto an an Abelian semiring factors through α.

THEOREM 9. *Let S be a normal semiring. Then the kernel of* α *is* S^1.

PROOF. If $x\alpha=0\alpha$, then $\sum a_i=0$, $\sum a_{i\sigma}=x$ for some $a_1, \cdots, a_n \in S$. But $\sum a_i=0$ clearly implies $a_i \in 0^*$ for each i. Hence $\sum a_{i\sigma}=x \in 0^*$. It is immediate that $x \in (0^*)^1=S^1$.

Hence every semiring in which 0^* is Abelian, and in particular, every semiring in which $0^*=0$ is semi-isomorphic to an Abelian semiring. Following [2], such semirings are called *conic*. It is not difficult to concoct examples which are not Abelian.

A similar difficulty arises in the Abelian case. Construct the free halfring $F(S)$ on a set S by taking the free Abelian (additive) semigroup with identity on the set of words in the free (multiplicative) semigroup on S, and defining multiplication distributively. Then every Abelian semiring is a homomorphic image of a free halfring, and by our first theorem, a semi-isomorphic image of a natural factor of the halfring. But it is easily seen that every natural factor of a halfring is a·halfring. Thus a cancellative semiring may have a non-cancellative semi-isomorphic image.

These weaknesses call forcible attention to the problem of when a semi-isomorphism must be an isomorphism. A partial answer is given in the following theorem.

THEOREM 10. *Let S be an intrinsically ordered semiring with positive cone P, and let* σ *be a semi-isomorphism of S such that* $S\sigma$ *is* $P\sigma$*-precancellative. Then the restriction of* σ *to any chain in S is one-to-one.*

PROOF. If x, y belong to a chain in S, we may assume $x \leq y$. Then $x+p=y$ for some $p \in P$, and $x\sigma+p\sigma=y\sigma$. If $x\sigma=y\sigma$, then $p\sigma \in Z(S\sigma) \cap P\sigma$, so that $p\sigma=0$. Then $p=0$ and σ is one-to-one.

COROLLARY. *Let S be a naturally linearly preordered semiring, and let* η *be a homomorphism of S onto a precancellative semiring. Then* $S\eta$ *is isomorphic to* $S/\mathrm{Ker}(\eta)$.

PROOF. By Theorem 1, they are semi-isomorphic. But $S/\mathrm{Ker}(\eta)$ is also naturally linearly preordered, and $S\eta$ is precancellative. Thus the semi-isomorphism is one-to-one, and thus an isomorphism.

A naturally preordered semiring S is linearly ordered provided that for each a, $b \in S$, one of the equations $a+x=b$, $b+x=a$ has a solution $x \in S$. This property has been powerfully exploited in [6] to obtain structural results. The good behavior of this class of semirings accounts for our interest in preordered semirings generally.

We now investigate the relationship of preordered semirings with ordered semirings. If S is any preordered set, the relation $\equiv$ defined by $x\equiv y$ if and only if $x\leq y$ and $y\leq x$ is an equivalence relation on S. If ω is the canonical mapping of S onto the set of equivalence classes, the relation $x\omega\leq y\omega$ if and only if $x\leq y$ is a partial order on $S\omega$, and ω is monotone. In the case of semirings we have

THEOREM 11. *Let S be an intrinsically preordered semiring with positive cone P. Then $S\omega$ is intrinsically ordered with positive cone $P\omega$, and ω is an order-epimorphism whose kernel is $St_p(0)$. If η is a monotone homomorphism of S onto an ordered semiring, then $\eta=\omega\phi$ for some monotone homomorphism ϕ of $S\omega$; and ϕ is an epimorphism when η is.*

COROLLARY. *A necessary condition for S to be ordered is that P be conic. If S is precancellative, this is sufficient.*

PROOF. *If $x\leq y\leq x$,* there are p, $q\in P$ such that $x+p=y$, $y+q=x$. Then $x+(p+q)=x$, and $p+q\in Z(S)=0$. If P is conic, $p=q=0$, and $x=y$.

If follows that a centric precancellative semiring S is naturally ordered if and only if it is conic. Since S is normal, 0^* is a stable ideal in S, and $S/0^*$ is conic. Thus 0^* behaves like a radical in the category of centric precancellative semirings: the "semisimple" structures are naturally ordered, while the "radical" structures are skewrings (additively groups). This is an unusual situation, in that the "radical" semirings are better behaved than the "semisimple" ones. In the case of a halfring S, 0^* is a ring, and $S/0^*$ is a conic halfring; i.e., a positive cone in its ring of differences. Thus the study of halfrings reduces to the study of rings, positive cones of partially ordered rings, and the halfring extension problem.

There is another reason why the category of centric precancellative semirings is attractive. It is easily seen that if K is an ideal centric in S, $x\equiv y$ (mod K) if and only if $x+K$, $y+K\neq\phi$. This is the same definition which has always been used in the Abelian case. Using this result, it is easy to show that K is the kernel of a homomorphism of a centric semiring S onto a precancellative (and necessarily centric) semiring if and only if K is hypersubtractive and centric in S.

We conclude by giving a decomposition theorem for certain homomorphisms of normal semirings. A homomorphism η of S onto T is an *epimorphism* if η maps

$S\backslash St_S(0)$ onto $T\backslash St_T(0)$. Let α, β be the natural homomorphisms of S onto $S/St_S(0)$, T onto $T/St_T(0)$ respectively, and define $\eta^o: S/St_S(0) \to T/St_T(0)$ by $(\beta\alpha)\eta_o = (S\eta)\beta$. Evidently η^o is an epimorphism. We also define $\eta^*: St_S(0) \to St_T(0)$ to be the restriction of η to $St_S(0)$; clearly η^* is an epimorphism. We have

THEOREM 12. *Let S be a normal semiring, and let η be an epimorphism of S. Then η^* and η^o are epimorphisms, and if η is an isomorphism, so are η^* and η^o. If S is centric and $S\eta$ precancellative, then η is an isomorphism if and only if both η^* and n^o are.*

PROOF. Let η^* and η^o be isomorphisms and let $x\eta = y\eta$, where S is centric, $S\eta$ precancellative. If $x \in St_S(0)$, then so must y, and $x\eta^* = x\eta = y\eta = y\eta^*$, so that $x = y$. If $x \notin St_S(0)$, neither does y, and $(x\alpha)\eta^o = (x\eta)\beta = (y\eta)\beta = (y\alpha)\eta^o$, so that $x\alpha = y\alpha$. Now $St_S(0)$ is centric if S is centric, and thus $x + r = y$ for some $r \in St_S(0)$. Thus $x\eta = y\eta = (x+r)\eta = x\eta + r\eta$, and $r\eta = 0$ since $S\eta$ is precancellative. But $r\eta = r\eta^* = 0$ implies $r = 0$, so that $x = y$.

Thus questions about homomorphisms of centrc precancellative semirings can be readily reduced to questions about homorphisms of conic semirings.

University of Pittsburgh
Pittsburgh, Pennsylvania 15213

REFERENCES

[1] S. Bourne, *On the homomorphism theorem for semirings*, Proc. Nat. Acad. Sci. U.S.A. 38 (1952), 119.

[2] L. Fuchs, *Partially Ordered Algebraic Systems*, Pergamon Press, London, 1963.

[3] K. Iizuka, *The Jacobson radical of a semiring*, Tohoku Math J. 2 (1959), 409-421.

[4] D.R. LaTorre, *On h-ideals and k-ideals in hemirings*, Publ. Math. Debrecen 12 (1965), 219-226.

[5] E.S. Ljapin, *Semigroups*, Translations of Mathematical Monographs Vol. 3, American Mathematical Society, Providence, 1963.

[6] J.R. Mosher, *Generalized quotients of hemirings*, Composctio Math. 22 (1970), 275-281

[7] K. Murata, *On the quotient semigroup of a non-commutative semigroup*, Osaka Math. J. 2 (1950), 1-5.

[8] H.E. Stone, *Structure Theory of Semirings*, Dissertation, Texas Christian University, Fort Worth, 1969.