Kyungpook Math. J.
Volume 13, Number 1
June, 1973

A NOTE ON A C-UMBILICAL HYPERSURFACE OF A 6-DIMENSIONAL K-SPACE

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The 6-dimensional K-space (or the almost Tachibana space) has been studied by Takamatsu [3] and Matsumoto [1]. The orientable hypersurface of a K-space admits an almost contact metric structure. We know that in an orientable C-umbilical hypersurface of a K-space, the induced structure tensor is an conformal Killing tensor (see [4]). On the other hand, it was proved in [2] that any conformal Killing tensor in a Riemannian manifold of constant curvature is decomposed into the form stated in the following theorem. And the decomposition of conformal Killing tensor in a Sasakian manifold was studied in [6].

In the present paper we study the C-umbilical hypersurface of a 6-dimensional K-space and prove the following main theorem

THEOREM. *In a C-umbilical hypersurface of a* 6-*dimensional K-space, the induced structure tensor* $f_j^{\ i}$ *is uniquely decomposed as follows*:

$$f_j^{\ i}=w_j^{\ i}+q_j^{\ i}.$$

if C-mean curvature $\alpha\neq-\left(1+\frac{k}{30}\right)$, *where* $w_j^{\ i}$ *is Killing tensor,* $q_j^{\ i}$ *is a closed conformal Killing tensor and k is a scalar curvature of the* 6-*dimensional K-space.*

1. Preliminalies.

Let $\widetilde{M}$ be an almost Hermitian manifold of dimension n (>2) with Hermitian structure (F_β^α, $G_{\beta\alpha}$), i.e. with an almost complex structure $F_\beta^{\ \alpha}$ and a positive definite Riemannian metric tensor $G_{\beta\alpha}$ satisfying

$$F_\beta^{\ \lambda}F_\lambda^{\ \alpha}=-\delta_\beta^{\ \alpha},\ G_{\lambda\mu}F_\beta^{\ \lambda}F_\alpha^{\ \mu}=G_{\beta\alpha}.$$

Putting $F_{\beta\alpha}=F_\beta^{\ \lambda}G_{\lambda\alpha}$, we have $F_{\beta\alpha}=-F_{\alpha\beta}$.

If an almost Hermitian manifold satisfies

$$\nabla_\beta F_\alpha^{\ \gamma}+\nabla_\alpha F_\beta^{\ \gamma}=0, \tag{1.1}$$

where ∇_β denotes the operator of covariant differentation with respect to the metric $G_{\beta\alpha}$, then the manifold is called a K-space, or an almost Tachibana space. Now let $K_{\gamma\beta\alpha\lambda}$, $K_{\beta\alpha}=K_{\lambda\beta\alpha}{}^{\lambda}$ and $K={}_{\beta\alpha}G^{\beta\alpha}$ be the Riemannian curvature, the Ricci tensor and scalar curvature respectively.

Let M be a orientable hypersurface of a K-space $\widetilde{M}$ and M be expressed by $X^\lambda=X^\lambda(x^h)$, where $\{X^\lambda\}$ and $\{x^h\}$ are local coordinates of $\widetilde{M}$ and M. If we put $B_i{}^\lambda=\partial_i X^\lambda$ ($\partial_i=\partial/\partial x^i$), then $B_i{}^\lambda$ are linear independent tangent vectors at each point of M. The induced Riemannian metric g_{ji} in M is given by $g_{ji}=G_{\beta\alpha}B_j^\beta B_i^\alpha$.

Choosing a unit normal C^λ to the hypersurface M in such way that C^λ and $B_i{}^\beta$ form a frame of positive orientation of $\widetilde{M}$ and $B_i{}^\beta$ form a frame of positive orientation of M, then we have

$$G_{\beta\alpha}B_i{}^\beta C^\alpha=0,\quad G_{\beta\alpha}C^\beta C^\alpha=1,$$

The transformations $F_\lambda{}^\alpha B_i{}^\lambda$ of $B_i{}^\lambda$ by $F_\lambda{}^\alpha$ and $F_\lambda{}^\alpha C^\lambda$ of C^λ by $F_\lambda{}^\alpha$ can be expressed as

$$(1.2)\qquad F_\lambda{}^\alpha B_i{}^\lambda=f_i{}^h B_h{}^\alpha+u_i C^\alpha,\quad F_\lambda{}^\alpha C^\lambda=-u^h B_h{}^\alpha,$$

where $u^h=g^{hi}u_i$, from which, we have

$$(1.3)\qquad f_i{}^h=B^h{}_\lambda F_\beta{}^\lambda B_i{}^\beta,\quad u_i=C_\lambda F_\beta{}^\lambda B_i{}^\beta=B_i{}^\beta F_{\beta\lambda}C^\lambda,$$

where $\qquad B_\beta{}^h=G_{\beta\alpha}B_j{}^\alpha g^{hj},\quad C_\beta=G_{\beta\alpha}C^\alpha.$

We can easily see that the set $(f_j{}^i,\ u^i,\ u_i,\ g_{ji})$ define an almost contact metric structure.

Denoting by ∇_j the operator of covariant differentiation with respect to the induced metric g_{ji}, we have the following Gauss and Weingarten equations for hypersurface respectively.

$$(1.4)\qquad \nabla_j B_i{}^\lambda=H_{ji}C^\lambda,\quad \nabla_j C_\lambda=-H_{ji}B^i{}_\lambda,$$

where H_{ji} is the second fundamental tensor of the hypersurface M. Differentiating (1.2) covariantely along M and taking account of (1.1) and (1.4), we have

$$(1.5)\qquad \nabla_j f_i{}^h+\nabla_i f_j{}^h=-2u^h H_{ji}+u_j H_i{}^h+u_i H_j{}^h,$$

$$\nabla_j u_i+\nabla_i u_j=-f_i^h H_{jh}-f_j^h H_{ih}$$

because B_i^λ, C^λ are linearly independant.

The hypersurface in which the second fundamental tensor has the form

$$H_{ji}=g_{ji}+\alpha u_j u_i \tag{1.6}$$

is called a C-umbilical hypersurface (see [5]). In C-umbilical hypersurface, (1.5) becomes

$$\begin{aligned}&\nabla_j f_i^{\ h}+\nabla_i f_j^{\ h}=-2u^h g_{ji}+u_j\delta_i^{\ h}+u_i\delta_j^{\ h},\\ &\nabla_j u_i+\nabla_i u_j=0.\end{aligned} \tag{1.7}$$

Thus *in a C-umbilical hypersurface of a K-space, the induced structure tensor* $f_j^{\ i}$ *is a conformal Killing tensor with associated vector* $-u^i$ *and induced structure vector* u^i *is a Killing vector* [4].

2. C-umbilical hypersurface of a 6-dimensional K-space.

In a 6-dimensional K-space $\widetilde{M}$,

$$K_{\beta\alpha}=\frac{K}{6}G_{\beta\alpha}, \tag{2.1}$$

that is, a 6-dimensional K-space $\widetilde{M}$ is an Einstein space [1].

LEMMA 2.1 [4] *In a 6-dimensional K-space,*

$$K_{\gamma\beta\alpha\lambda}-K_{\gamma\beta\mu\nu}F_\alpha^{\ \mu}F_\lambda^{\ \nu}=e(G_{\gamma\lambda}G_{\beta\alpha}-G_{\gamma\alpha}G_{\beta\lambda}-F_{\gamma\lambda}F_{\beta\alpha}+F_{\gamma\alpha}F_{\beta\lambda}), \tag{2.2}$$

where $e=\frac{k}{30}$ *is a positive constant.*

Substituting (2.2) into the Gauss and Codazzi equations:

$$R_{kjih}=B_k^{\ \nu}B_j^{\ \mu}B_i^{\ \lambda}B_h^{\ \gamma}K_{\nu\mu\nu\gamma}+H_{kh}H_{ji}-H_{jh}H_{ki}, \tag{2.3}$$

$$\nabla_k H_{ji}-\nabla_j H_{ki}=B_k^{\ \nu}B_j^{\ \mu}B_i^{\ \lambda}C^\gamma K_{\nu\mu\lambda\gamma}, \tag{2.4}$$

we have

$$\begin{aligned}R_{kjih}=&H_{kh}H_{ji}-H_{ki}H_{jh}+\{R_{kjlm}-(H_{km}H_{jl}\\ &-H_{kl}H_{jm})\}f_i^{\ l}f_h^{\ m}+e(g_{kh}g_{ji}-g_{ki}g_{jh}-f_{kh}f_{ji}+f_{ki}f_{jh}),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\nabla_k H_{ji}-\nabla_j H_{ki}=&(\nabla_k H_{jm}-\nabla_j H_{km})u_i u^m\\ &+\{(H_{km}H_{jl}-H_{jm}H_{kl})-R_{kjlm}\}f_i^{\ l}u^m+e(f_{ki}u_j-f_{ji}u_k).\end{aligned} \tag{2.6}$$

Substituting (1.6) into (2.4) and transvecting with g^{ih}, we have

$$\nabla_j\alpha=\lambda u_j, \tag{2.7}$$

where $\lambda=u^i\nabla_i\alpha$. Operating ∇_k to (2.7), we have

$$\nabla_k\nabla_j\alpha=(\nabla_k\lambda)u_j+\lambda\nabla_k u_j. \tag{2.8}$$

Interchanging the indices j and k in (2.8) and subtracting the equation thus obtained from (2.8), we have

(2.9) $$(\nabla_k\lambda)u_j-(\nabla_j\lambda)u_k+\lambda(\nabla_k u_j-\nabla_j u_k)=0,$$

because $\nabla_j\alpha$ is a gradiant vector. Transvecting (2.9) with f^{kj} and taking account of that u^i is a Killing vector, we have

$$\lambda f^{kj}\nabla_k u_j=0.$$

Hence $\lambda=0$ because $f^{kj}\nabla_k u_j=4$ in C-umbilical hypersurface of 6-dimensional K-space [4]. Thus we have

LEMMA 2.2. *In C-umbilical hypersurface of a 6-dimensional K-space, the C-mean curvature α is a constant.*

Substituting (1.6) into (2.5), we have

(2.10) $$\begin{aligned}R_{kjih}=&g_{kh}g_{ji}-g_{ki}g_{jh}+\alpha(g_{kh}u_ju_i+g_{ji}u_ku_h\\&-g_{ki}u_ju_h-g_{jh}u_ku_i)+\{R_{kjlm}-(g_{km}g_{jl}-g_{kl}g_{jm})\\&-\alpha(g_{km}u_ju_l+g_{jl}u_ku_m-g_{kl}u_ju_m-g_{jm}u_ku_l)\}f_i^lf_h^m\\&+e(g_{kh}g_{ji}-g_{ki}g_{jh}-f_{kh}f_{ji}+f_{ki}f_{jh}).\end{aligned}$$

Transvecting (2.10) with u^i and taking account of $f_j^iu^j=0$, we have

(2.11) $$R_{kjih}u^i=C(u_jg_{kh}-u_kg_{jh}),$$

where C is a constant by Lemma 2.2.

On the other hand u^i is a Killing vector, we have

$$\nabla_h\nabla_k u_j+R_{kjih}u^i=0,$$

from which,

(2.12) $$\nabla_h\nabla_k u_j=C(u_kg_{jh}-u_jg_{kh}).$$

3. Proof of the main Theorem.

LEMMA 3.1. *In a C-umbilical hypersurface of a 6-dimensional K-space, if $\nabla_k p_{ji}=0$ for a skew symmetric tensor p_{ji} and $\alpha\neq-(1-e)$, then $P_{ji}=0$.*

PROOF. By Ricci identity, we have

(3.1) $$R_{khj}{}^lP_{li}+R_{khi}{}^lP_{jl}=0.$$

Contracting (3.1) with u^hu^j and $g^{hj}u^k$ respectively and using (2.11), we have

$$C(P_{ki}-u_k P_i+u_i P_k)=0, \qquad 3CP_i=0,$$

where $P_i=u^l P_{li}$.

From the above two equations we have $P_{ji}=0$.

Proof of the Theorem. From (1.7) and (2.12), we have

$$\nabla_j(Cf_{ih}-\nabla_i u_h)+\nabla_i(Cf_{jh}-\nabla_j u_h)=0.$$

Putting $Cf_{ih}-\nabla_i u_h=u_{ih}$, then the skew symmetric tensor u_{ih} is a Killing tensor. Therefore we have

$$f_{ih}=w_{ih}+q_{ih},$$

where $w_{ih}=(1/C)u_{ih}$ is a Killing tensor and $q_{ih}=(1/C)\nabla_i u_h$ is a closed conformal Killing tensor. Next we prove the uniqueness of the decomposition. If there exist a Killing tensor w'_{ih} and a closed conformal Killing tensor q'_{ih} such that $f_{ih}=w'_{ih}+q'_{ih}$, then $w_{ih}-w'_{ih}=q'_{ih}-q_{ih}$ is a closed Killing tensor. Thus

$$\nabla_j(w_{ih}-w'_{ih})=0, \ \nabla_j(q'_{ih}-q_{ih})=0.$$

Therefore $w_{ih}=w'_{ih}$, $q_{ih}=q'_{ih}$ by Lemma 3.1. This completes the proof of Theorem.

When the hypersurface is a totally umbilical, we have

$$g_{ji}\nabla_k H-g_{ki}\nabla_j H=K_{\nu\mu\lambda\gamma}B_k^{\ \nu}B_j^{\ \mu}B_i^{\ \lambda}C^{\gamma},$$

where $H=g^{ji}H_{ji}/5$ is a mean curvature. Transvecting the equation above with g^{ji} and using (2.1), we see that the mean curvature H is a constant. Thus

COROLLARY. *In a totally umbilical hypersurface of 6-dimensional K-space, the induced structure tensor is decomposed as follows:*

$$f_j^{\ i}=w_j^{\ i}+q_j^{\ i},$$

where $w_j^{\ i}$ is a Killing tensor and $q_j^{\ i}$ is a closed conformal Killing tensor.

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