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ON A CHARACTERIZATION OF SPACES OF CONSTANT HOLOMORPHIC CURVATURE IN TERMS OF GEODESIC HYPERSPHERE

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Introduction. Let M^n be a Riemannian space of positive definite metric and 0 its point. We denote by s the geodesic distance from 0. Then M^n is called harmonic at 0 if the Laplacian Δs is a function of s only in a neighbourhood of 0. If M^n is harmonic at any point, it is called harmonic. M^n is harmonic at 0 if and only if the mean curvature of each geodesic hypersphere of center 0 is constant (Cf. §2). A space of constant curvature and a space of constant holomorphic curvature are examples of harmonic Riemannian spaces. Thus the geodesic hypersphere in these spaces is expected to have more special properties. The discussions in this paper are local and the differentiability is of C^∞. As to notations we follow Yano-Bochner [1] with trivial changes.

1. Normal coördinates. Consider an n dimensional Riemannian space M^n with positive definite metric g_{ij}. Let 0 be a point of M^n and $\{x^i\}$ a normal coordinate of origin 0. The coordinate $\{x^i\}$ is an allowable coordinate satisfying

$$g_{ij}x^j=(g_{ij})_0x^j, \tag{1.1}$$

where g_{ij} and $(g_{ij})_0$ denote the metric tensors at (x^i) and 0 respectively, [2]. Let U be a coordinate neighbourhood where $\{x^i\}$ is valid. A curve in U which goes through 0 is a geodesic if and only if it is written as

$$x^i=\xi^i s \tag{1.2}$$

where s is the arc length measured from 0 and ξ^i are constant such that

$$(g_{ij})_0\xi^i\xi^j=1. \tag{1.3}$$

We take U so small that any point (x^i) in U is connected with 0 by a unique geodesic in U. Then any point in U has the representation (1.2) with ξ^i and s as parameters, and conversely ξ^i and s are regarded as functions of (x^i).

From (1.2) and (1.3) it follows that

$$(g_{ij})_0 x^i x^j = s^2.$$

Operating $\partial_k = \partial/\partial x^k$ to this equation and putting $s_k = \partial_k s$, we have

$$(1.4) \qquad (g_{ik})_0 x^i = s s_k.$$

Hence $s s_k = g_{ik} x^k$ follows by virture of (1.1) and we can obtain

$$(1.5) \qquad s^i s_i = 1, \quad (s^i = g^{ik} s_k).$$

The last equation shows that s^i is a unit vector field in U with singularity at 0.

By the covariant differentiation of (1.5) we have

$$(1.6) \qquad s^i \nabla_j s_i = 0.$$

Now, let $\{x^i\}$ under consideration satisfy

$$(1.7) \qquad (g_{ij})_0 = \delta_{ij}.$$

It is known that such a normal coordinate always exists. We shall use ξ_i instead of ξ^i in such a coordinate.

From (1.4) and (1.7) we have

$$(1.8) \qquad x^k = \xi_k s = s s_k,$$

$$(1.9) \qquad \xi_k = s_k. \qquad (s \neq 0).$$

Operating ∂_j to (1.8) we get

$$\delta_{jk} = \xi_k s_j + s \partial_j \xi_k$$

and taking account of (1.9) we obtain

$$(1.10) \qquad \partial_j \xi_k = \partial_j s_k = \frac{1}{s} \Delta_{jk},$$

where

$$(1.11) \qquad \Delta_{jk} = \delta_{jk} - \xi_j \xi_k = \delta_{jk} - s_j s_k.$$

It is easy to see that the following equations

$$(1.12) \qquad \begin{aligned} &\Delta_{ij} = \Delta_{ji}, \quad \Delta_{ij} \xi_j = 0, \\ &\Delta_{ij} \Delta_{jk} = \Delta_{ik}, \quad \Delta_{ii} = n-1 \end{aligned}$$

are valid, where the repeated indices are taken to sum from 1 to n.

Consider a space M^n of constant curvature with the sectional curvature $k = \pm l^2$, l being a positive constant. Let 0 be any point and take it as the origin of a normal coordinate satisfying (1.7). Then it is known [3] that the metric tensor g_{ij} at $x^i = \xi_i s$ is given by

(1.13) $$g_{ij}=\xi_i\xi_j+\gamma\Delta_{ij},$$

where γ is a function of s defined by

(1.14) $$\gamma(s)=\begin{cases}\left(\dfrac{\sin(ls)}{ls}\right)^2, & \text{if } k=l^2,\\[2mm] \left(\dfrac{\sinh(ls)}{ls}\right)^2, & \text{if } k=-l^2.\end{cases}$$

If we denote $d\gamma/ds$ by γ', the Christoffel symbols are

(1.15) $$\left\{{i \atop jk}\right\}=\left(\frac{1-\gamma}{s}-\frac{\gamma'}{2}\right)\xi_i\Delta_{jk}+\frac{\gamma'}{2\gamma}(\xi_j\Delta_{ik}+\xi_k\Delta_{ij})$$

and we can get the following equation:

(1.16) $$\nabla_k s_j=\left(\frac{\gamma}{s}+\frac{\gamma'}{2}\right)\Delta_{jk}.$$

2. The geodesic hypersphere. Consider a hypersurface M^{n-1} in a Riemannian space M^n and let $x^i=x^i(u^1, \cdots, u^{n-1})$ be its local expression, where $\{x^i\}$ and $\{u^a\}$ denote local coordinates in M^n and M^{n-1} respectively. We make a convention that Latin indices $a, b, \cdots$ take values from 1 to $n-1$. If we put $B_a^{\ i}=\partial x^i/\partial u^a$, then the induced Riemannian metric $'g_{ab}$ of M^{n-1} is written as

$$'g_{ab}=B_a^{\ i}B_b^{\ j}g_{ij}.$$

If we write by N^i the unit normal (local) vector field, then N^i and $N_i=g_{ij}N^j$ satisfy

$$N^iN_i=1, \quad B_a^{\ i}N_i=0.$$

Denoting the inverse matrix of $('g_{ab})$ by $('g^{ab})$, let us put

$$B^b_{\ j}={}'g^{ab}B_a^{\ i}g_{ij}.$$

It is known that the matrix $(B^a_{\ i}, N_i)$ is the inverse matrix of $(B_a^{\ i}, N^i)$ and hence

(2.1) $$B^a_{\ j}B_a^{\ i}=\delta_j^i-N_jN^i$$

hold good.

Let ∇_a be the operator of (generalized) covariant differentiation along M^{n-1}. By definition, we have

$$\nabla_aB_b^{\ i}=\partial_cB_b^{\ i}-'\left\{{c \atop ab}\right\}B_c^{\ i}+B_a^{\ j}B_b^{\ k}\left\{{i \atop jk}\right\}.$$

If X_i is a covariant vector field in M^n, it holds on M^{n-1} that $\nabla_aX_i=B_a^{\ j}\nabla_jX_i$. The Euler-Schouten tensor $H_{ab}^{\ \ i}$ is defined by $H_{ab}^{\ \ i}=\nabla_aB_b^{\ i}$. As is well known, there exists a tensor H_{ab} such that

(2.2) $$H_{ab}{}^{i}=H_{ab}N^{i}.$$

H_{ab} is the second fundamental tensor of M^{n-1} and the mean curvature H is the scalar function defined by

$$H=\frac{1}{n-1}{}'g^{ab}H_{ab}.$$

If $H_{ab}=H'g_{ab}$ is valid identically, M^{n-1} is called totally umbilic. If H_{ab} vanishes identically, then M^{n-1} is called totally geodesic.

Now, let 0 be a point of M^{n} and U a domain where a normal coordinate $\{x^{i}\}$ of origin 0 is valid. Suppose that ε is a positive constant so small that the set S_{ε} of points $(\xi^{i}\varepsilon)$ is contained in U. S_{ε} is called a geodesic hypersphere at 0 of radius ε.

As S_{ε} is a hypersurface, if $\{u^{a}\}$ denotes a local coordinate in S_{ε},

$$s(x^{i}(u^{a}))=\varepsilon$$

holds good on S_{ε} locally. Operating ∇_{a} to this equation we have

(2.3) $$B_{a}^{i}s_{i}=0.$$

Thus s^{i} is a normal vector field of S_{ε}, and by (1.5) it being unit we may take N_{i} as

(2.4) $$N_{i}=-s_{i}.$$

If we operate ∇_{b} to (2.3) and take account of (2.2) and (2.4), we can get

(2.5) $$H_{ab}=B_{a}{}^{i}B_{b}{}^{j}\nabla_{i}s_{j}$$

which is the key equation in this paper.

From (2.5) it follows that

$$(n-1)H=\Delta s\ (=g^{ij}\nabla_{i}s_{j})$$

by virtue of (2.1), (2.4) and (1.6). Consequently, we know that a Riemannian space is harmonic at 0 if and only if the mean curvature of each geodesic hypersphere at 0 is constant.

3. Spaces of constant curvature. Let M^{n} be a locally flat Riemannian space and 0 any point of M. Then there exists a normal coordinate system $\{x^{i}\}$ of origin 0 such that $g_{ij}=\delta_{ij}$. Consider a geodesic hypersphere S_{ε} at 0 of a small radius ε. The induced metric of S_{ε} is

$$(3.1)\qquad 'g_{ab}=B_a^{\ i}B_b^{\ j}\delta_{ij}=B_a^{\ i}B_b^{\ i}.$$

As we have

$$\nabla_i s_j=\partial_i s_j=\frac{1}{s}\Delta_{ij}$$

by (1.10), it follows that

$$B_a^{\ i}B_b^{\ j}\nabla_i s_j=\frac{1}{s}B_a^{\ i}B_b^{\ j}\Delta_{ij}=\frac{1}{s}\ 'g_{ab}$$

taking account of (2.3) and (3.1). Thus (2.5) gives $H_{ab}=\frac{1}{s}\ 'g_{ab}$ which shows that S_ε is totally umbilic.

Next, let M^n be a space of constant curvature K ($\neq 0$). we take a point 0 arbitrary and consider a normal coordinate $\{x^i\}$ of origin 0 satisfying (1.7). Then, for any geodesic hypersphere S_ε at 0, we have

$$H_{ab}=B_a^{\ i}B_b^{\ j}\nabla_i s_j=\left(\frac{\gamma}{s}+\frac{\gamma'}{2}\right)B_a^{\ i}B_b^{\ j}\Delta_{ij}$$

by virtue of (1.16). On the other hand, the induced metric of S_ε is

$$'g_{ab}=B_a^{\ i}B_b^{\ j}g_{ij}=B_a^{\ i}B_b^{\ j}(\xi_i\xi_j+\gamma\Delta_{ij})=\gamma B_a^{\ i}B_b^{\ j}\Delta_{ij}$$

on taking account of (1.9) and (2.3). Hence we obtain

$$H_{ab}=\left(\frac{1}{s}+\frac{\gamma'}{2\gamma}\right)\ 'g_{ab}$$

which proves the following

THEOREM 1. *In a space of constant curvature, each geodesic hypersphere at any point is totally umbilic.*

4. The converse problem. Consider an Einstein space M^n ($n>2$) and we assume that each geodesic hypersphere at any point is totally umbilic. Let 0 be any point of M^n and $\{x^i\}$ a normal coordinate of origin 0. As each geodesic hypersphere S_ε is totally umbilic,

$$(4.1)\qquad H_{ab}=\alpha\ 'g_{ab}=\alpha B_a^{\ j}B_b^{\ i}g_{ji}$$

holds good on each S_ε. If we substitute (2.5) into (4.1), it follows that

$$B_a^{\ j}B_b^{\ i}(\nabla_j s_i-\alpha g_{ji})=0,$$

and taking account of (2.1) we obtain

$$(4.2)\qquad \nabla_j s_i=\alpha(g_{ji}-s_j s_i).$$

Thus it follows

$$g^{ji}\nabla_j s_i=(n-1)\alpha,$$

from which we know that α is a function of (x^i).

If we differentiate (4.2) covariantly, we have

$$\nabla_k\nabla_j s_i=\alpha_k(g_{ji}-s_js_i)-\alpha^2(g_{kj}s_i+g_{ki}s_j-2s_ks_js_i),$$

where $\alpha_k=\partial_k\alpha$. Substituting the last equation into the Ricci's identity:

$$(4.3)\qquad \nabla_k\nabla_j s_i-\nabla_j\nabla_k s_i=-s_rR^r{}_{ijk},$$

we get

$$(4.4)\qquad -s_rR^r{}_{ijk}=\alpha_k(g_{ji}-s_js_i)-\alpha_j(g_{ki}-s_ks_i)-\alpha^2(g_{ki}s_j-g_{ji}s_k).$$

Transvecting (4.4) with g^{ji} we have

$$(4.5)\qquad -s_rR^r{}_k=(n-2)\alpha_k+(s^i\alpha_i)\,s_k+(n-1)\alpha^2s_k.$$

On the other hand, the Ricci tensor is of the form $R^r{}_k=(n-1)k\delta_k{}^r$ by the assumption, where k is a constant. Hence (4.5) becomes

$$(4.6)\qquad (n-2)\alpha_k+(s^i\alpha_i)s_k+(n-1)(\alpha^2+k)s_k=0.$$

Multiplying s^k and taking account of (1.5) we have

$$s^i\alpha_i=-(\alpha^2+k),$$

and substituting this equation into (4.6) we get

$$\alpha_k=-(\alpha^2+k)s_k.$$

Thus we obtain from (4.4)

$$(4.7)\qquad s_rR^r{}_{ijk}=k(g_{ji}s_k-g_{ki}s_j).$$

Now, let $Z^h{}_{ijk}$ be the concircular curvature tensor defined by

$$Z^h{}_{ijk}=R^h{}_{ijk}-k(\delta_k{}^hg_{ij}-\delta_j{}^hg_{ki}),$$

then (4.1) is written as $s_rZ^r{}_{ijk}=0$ or

$$(4.8)\qquad x^hZ_{hijk}=0.$$

Let us consider a geodesic $x^h=\xi^hs$. From (4.8) $\xi^hZ_{hijk}=0$ are valid on the geodesic except 0 and by the continuity we get $\xi^h(Z_{hijk})_0=0$. As ξ^h are arbitrary and 0 is any, we know that Z_{hijk} vanishes identically. Thus M^n is of constant curvature and hence we have proved the following

THEOREM 2. *In an $n(>2)$ dimensional Einstein space M^n, if each geodesic hypersphere at any point is totally umbilic, then M^n is a space of constant curvature.*

5. η-umbilic hypersurfaces in a Kählerian space. Let M^{2m} be a Kählerian space, i.e., a $2m(=n)$ dimensional Riemannian space which admits a parallel tensor $F_i^{\ h}$ called the complex structure such that

$$F_i^{\ r}F_r^{\ h}=-\delta_i^h, \qquad F_i^{\ r}F_j^{\ s}g_{rs}=g_{ij}.$$

Consider a hypersurface M^{2m-1} in M^{2m} and let N^i its unit normal (local) vector field. If we define η_a by

$$\eta_a=B_a^{\ i}F_i^{\ h}N_h, \tag{5.1}$$

it has a meaning over M^{2m-1} within sign. M^{2m-1} is called totally η-umbilic if

$$H_{ab}=\alpha' g_{ab}+\beta\eta_a\eta_b$$

holds good for some scalar functions α and β.

A Kählerian space M^{2m} is called constant holomorphic curvature, if its curvature tensor satisfies

$$R_{hijk}=k(g_{hk}g_{ij}-g_{hj}g_{ik}+F_{hk}F_{ik}-F_{hj}F_{ik}-2F_{hi}F_{jk}), \text{ where } F_{ij}=F_i^{\ r}g_{rj}.$$

In this section we prove the following

THEOREM 3. *In a space of constant holomorphic curvature, each geodesic sphere at any point is totally η-umbilic.*

PROOF. Let 0 be a point in a space of constant holomorphic curvature M^{2m} of non-zero k. It is known that M^{2m} admits an allowable complex coordinate $\{z^\lambda\}$ of origin 0 such that the metric tensor is given by

$$g_{\alpha\beta^*}=\frac{1}{S^2}(S\delta_{\alpha\beta}-2kz^{\alpha^*}z^\beta), \qquad g_{\alpha\beta}=g_{\alpha^*\beta^*}=0,$$

where z^{α^*} means $\bar z^\alpha$, the complex conjugate of z^α, and

$$S=1+2ku, \qquad u=z^\alpha z^{\alpha^*}. \tag{5.2}$$

(Greak indices $\alpha, \beta, \cdots, \lambda, \mu, \cdots$ run from 1 to m, and $\alpha^*=\alpha+m$. $u=z^\alpha z^{\alpha^*}$ means $\sum z^\alpha z^{\alpha^*}$). g^{ij} are given with respect to $\{z^\lambda\}$ by

$$g^{\alpha\beta^*}=S(\delta^{\alpha\beta}+2kz^\alpha z^{\beta^*}), \quad g^{\alpha\beta}=g^{\alpha^*\beta^*}=0.$$

The Christoffel symbols are zero except

$$\Gamma_\beta{}^\alpha{}_\gamma=-\frac{2k}{S}(\delta_\gamma^{\ \alpha}z^{\beta^*}+\delta_\gamma^{\ \beta}z^{\alpha^*}) \tag{5.3}$$

and their complex conjugates $\Gamma_{\beta^*}{}^{\alpha^*}{}_{\gamma^*}$.

Now, l being a positive constant, let us put

$$k=\begin{cases} l^2, & \text{if } k>0, \\ -l^2, & \text{if } k<0. \end{cases}$$

Then it is known [4] that in a neighbourhood of any 0 point (z^λ) is represented as

(5.4) $$z^\lambda=A^\lambda\tan(ls), \text{ or}$$
$$z^\lambda=A^\lambda\tan(ls)$$

according as $k>0$ or $k<0$, where s denotes the distance from 0 to (z^λ) and A^λ are complex numbers such that

(5.5) $$2l^2A^\lambda\overline{A}^\lambda=1.$$

Henceforce we shall consider only the case $k>0$, because the calculation of the case $k<0$ is similar.

From (5.2), (5.4) and (5.5) we have

$$S=1+2l^2z^\lambda z^{\lambda*}=\sec^2(ls),$$

and by differentiation with respect to z^α we get

(5.6) $$z^{\alpha*}=\lambda s_\alpha,$$

where we have put

$$\lambda=\frac{1}{l}\sec^2(ls)\tan(ls)=\frac{S}{l}\tan(ls).$$

From (5.6) it follows that

$$\partial_\beta s_\alpha=-s_\alpha\partial_\beta\log\lambda,$$
$$\partial_{\beta*}s_\alpha=\frac{1}{\lambda}\delta_{\alpha\beta}-s_\alpha\partial_{\beta*}\log\lambda.$$

Substituting into the last equations

$$\partial_\beta\log\lambda=\frac{1}{\tan(ls)}(3S-2)s_\beta,$$

we have

$$\partial_\beta s_\alpha=-\mu(3S-2)s_\alpha s_\beta,$$
$$\partial_{\beta*}s_\alpha=\mu\{\frac{1}{S}\delta_{\alpha\beta}-(3S-2)s_\alpha s_{\beta*}\},$$

where μ is defined by

$$\mu=\cot(ls).$$

Thus we can get taking account of (5.3) and (5.6)

(5.7) $$\nabla_\beta s_\alpha=\mu(S-2)s_\alpha s_\beta,$$
$$\nabla_{\beta*}s_\alpha=\mu\left\{\frac{1}{S}\delta_{\alpha\beta}-(3S-2)s_\alpha s_{\beta*}\right\}.$$

Now we consider the real coordinate $\{x^i\}$ which is associated to $\{z^\lambda\}$ by $z^\lambda=x^\lambda$

$+ix^{\lambda*}$. Then (5.7) is written with respect to $\{x^i\}$ as

(5.8) $$\nabla_i s_j = \mu(g_{ij} - s_i s_j) + \nu \tilde{s}_i \tilde{s}_j,$$

where we have put

$$\nu = -l \tan(ls), \quad \tilde{s}_i = F_i^{\ r} s_r.$$

As (5.8) is a tensor equation, it is still valid in any allowable coordinate $\{x^i\}$. Hence we may regard $\{x^i\}$ as a normal coordinate of origin 0. Let S_ε be a geodesic hypersphere at 0 and we follow the notations in §2. From (2.5) and (5.8) we have

(5.9) $$H_{ab} = B_a^{\ i} B_b^{\ j} \nabla_i s_j = \mu' g_{ab} + \nu \eta_a \eta_b$$

taking account of (1.6) and (5.1). Consequently S_ε is totally η-umbilic, which proves the theorem.

We remark that μ and ν in (5.9) satisfy $\mu\nu = -k$.

6. The converse problem. In this section we shall see the following theorem to be valid.

THEOREM 4. *In an Einstein-Kählerian space* M^{2m} $(m>1)$, *if each geodesic hypersphere at any point* 0 *satisfies*

$$H_{ab} = \mu' g_{ab} + \nu \eta_a \eta_b$$

for some functions μ *and* ν *such that* $\mu\nu$=*constant, then* M^{2m} *is a space of constant holomorphic curvature.*

PROOF. Denoting by $\{x^i\}$ a normal coordinate of origin 0, we follow the notations in §2. For each geodesic hypersphere at 0, we have from (2.5) and the assumption

(6.1) $$B_a^{\ j} B_b^{\ i} (\nabla_j s_i - \mu g_{ji} - \nu \tilde{s}_j \tilde{s}_i) = 0.$$

Transvecting (6.1) with $B^a_{\ k} B^b_{\ l}$ and taking account of $F_{ij} = -F_{ji}$ and (1.6), we get

(6.2) $$\nabla_j s_i = \mu(g_{ji} - s_j s_i) + \nu \tilde{s}_j \tilde{s}_i.$$

As (6.2) holds good on each geodesic hypersphere at 0, we can easily see that μ and ν are functions of (x^i) in a neighbourhood of 0. If we differentiate (6.2) covariantly and make use of (6.2) itself, it follows that

$$\nabla_k \nabla_j s_i = \mu_k (g_{ij} - s_i s_j) - \mu^2 (s_j g_{ki} + s_i g_{kj} - 2 s_k s_j s_i)$$

$$+\nu_k\tilde{s}_j\tilde{s}_i-\nu^2\tilde{s}_k(s_j\tilde{s}_i+\tilde{s}_js_i)$$
$$-\mu\nu(\tilde{s}_ks_j\tilde{s}_i+\tilde{s}_k\tilde{s}_js_i-F_{jk}\tilde{s}_i-F_{ik}\tilde{s}_j+2s_k\tilde{s}_j\tilde{s}_i).$$

Substituting this equation into the Ricci's identity (4.3), we have

$$\begin{aligned}(6.3)\qquad -s_rR^r{}_{ijk}&=\mu_k(g_{ij}-s_is_j)-\mu_j(g_{ik}-s_is_k)\\ &-\mu^2(s_jg_{ki}-s_kg_{ji})+\tilde{s}_i(\nu_k\tilde{s}_j-\nu_j\tilde{s}_k)\\ &-\nu^2\tilde{s}_i(\tilde{s}_ks_j-\tilde{s}_js_k)\\ &+2\mu\nu F_{jk}\tilde{s}_i+F_{ik}\tilde{s}_j-F_{ij}\tilde{s}_j-\tilde{s}_i(\tilde{s}_js_k-\tilde{s}_ks_j).\end{aligned}$$

If we transvect (6.3) with g^{ij} and take account of

$$(6.4)\qquad R_k^r=C\delta_k^r,$$

C being a constant, it follows that

$$(6.5)\qquad -Cs_k=(n-2)\mu_k+\{s^i\mu_i+(n-1)\mu^2+\nu^2+2\mu\nu\}s_k+\nu_k-\tilde{s}_i\nu^i\tilde{s}_k.$$

Multiplying (6.5) by s^k, we get

$$(6.6)\qquad -C=(n-1)(s^i\mu_i+\mu^2)+\nu^2+2\mu\nu+s^i\nu_i$$

and multiplying $\tilde{s}^k=g^{kj}\tilde{s}_j$ to (6.5) we have

$$(6.7)\qquad (n-2)\mu_k\tilde{s}^k=0.$$

On the other hand, it is known that the curvature tensor of a Kählerian space satisfies

$$(6.8)\qquad 2F_i^tR_t^r=F^{jk}R^r{}_{ijk}.$$

Hence if we transvect (6.3) with F^{jk} and make use of (6.8), (6.7) and (6.4), it follows that

$$-C\tilde{s}_i=\tilde{\mu}_i+(s^l\nu_l+\mu^2+\nu^2+n\mu\nu)\tilde{s}_i.$$

Thus we get

$$(6.9)\qquad \mu_i=-(C+s^l\nu_l+\mu^2+\nu^2+n\mu\nu)s_i,$$

from which it follows that

$$(6.10)\qquad -C=s^i\mu_i+s^i\nu_i+\mu^2+\nu^2+n\mu\nu.$$

From (6.6) and (6.10) we can get

$$s^i\mu_i=\mu(\nu-\mu),$$
$$s^i\nu_i=-C-(n+1)\mu\nu-\nu^2.$$

Consequently we have from (6.9)

$$\mu_i=\mu(\nu-\mu)s_i \tag{6.11}$$

and

$$\nu_i=-\nu(\nu-\mu)s_i \tag{6.12}$$

taking account of $\mu\nu$=constant. The substitution of (6.11) and (6.12) into (6.3) finally shows that

$$s_r U^r{}_{ijk}=0 \tag{6.13}$$

are valid, where we have put

$$U^r{}_{ijk}=R^r{}_{ijk}+\mu\nu(\delta^r_k g_{ji}-\delta^r_j g_{ki}+F_k{}^r F_{ji}-F_j{}^r F_{ki}-2F_{kj}F_i{}^r).$$

Now we apply the similar process as in §4 to (6.13) and complete the proof.

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