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EPIREFLECTIVE SUBCATEGORIES OF PARTIALLY ORDERED TOPOLOGICAL SPACES

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Introduction and Notation: In this paper we consider the categories **PTop** and **HOTS** of all partially ordered topological spaces, and all continuously partially ordered Housdorff spaces, respectively, with the continuous, isotone functions as their morphisms.

Let Top be the category of topological spaces, **H** of Hausdorff spaces, **Cr** of completely regular spaces and **C** of compact (and Hausdorff) spaces. For a full subcategory **K** of **Top**, we denote by **KPTop** the full subcategory of **PTop** consisting of all the objects whose underlying space belongs to **K**. Similarly we denote by **KOTS** the full subcategory of **HOTS** consisting of all objects whose underlying space belongs to **K**.

Firstly, we show that **PTop** is a complete, locally and colocally small category and find some important epireflective subcategories of **PTop**. The subcategories **KOTS** of **HOTS** which are epireflective are exactly those for which **K** is epireflective in **H**.

Secondly, introducing the category of all completely regular ordered topological spaces [7] and its subcategory **CrORR**, of all such spaces as defined in [7], we show that every object in **CrORR** can be characterized as a subspace of an object in COTS, and that **CrORR** is epireflective in **PTop**.

Finally, we consider the sets I_1X and C_1X of all continuous, isotone functions from X into the unit interval of the reals and into the reals, respectively. For $X\in$**CrORR** we define β_1X and υ_1X in the same way as the Stone-Cech compactification and Realcompactificaton of X, respectively.

1. General Epireflexions.

Let X be a set, $(Y_i)_{i\in I}$ a family of objects of **PTop**, and $f_i:X\to Y_i$ a family of functions which separates the points of X. Then X has an initial **PTop** structure with respect to $(f_i)_{i\in I}$. Indeed, we consider the initial topology on X

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with respect to the family $(f_i)_{i\in I}$ and define a partial order on X by $a\leqslant_X b$ if and only if $f_i(a)\leqslant f_i(b)$ for all $i\in I$. Clearly $\leqslant_X$ is a partial order, and one easily sees that $(X, \leqslant_X)$ has the initial **PTop** structure with respect to $(f_i)_{i\in I}$.

The proof for our next lemma follows directly from the above remark.

LEMMA 1. *If $(X_i)_{i\in I}$ is a family in* **PTop**, *there exists a product $\Pi_{i\in I}X_i\in$* **PTop** *and if $X\in$* **PTop** *and S is a subset of X, the inclusion $i:S\to X$ induces a* **PTop**-*subspace of X.*

LEMMA 2. *The category* **PTop** *is complete. Moreover, if* **K** *is a productive, closed hereditary subcategory of* **Top**, **KPTop** *is complete.*

PROOF. Considering [4], it is sufficient to show that **PTop** has equalizers. Given $X\underset{g}{\overset{f}{\rightrightarrows}}Y$ in **PTop**, the subspace K of X induced on $\{x\in X\,|\,f(x)=g(x)\}$ by its inclusion map $i:K\to X$ is a **PTop**-equalizer of f, g as one can easily check.

LEMMA 3. *Let $(X_i, \leqslant_i)_{i\in I}$ be a family of* **PTop**-*spaces. On the underlying set of the topological space $\amalg_{i\in I}X_i$ we difine the following partial order: $(x,i)\leqslant(y,j)$ if and only if $x\leqslant_i y$ and $i=j$. Then $(\amalg_{i\in I}X_i, \leqslant)$ is the coproduct $\amalg_{i\in I}(X_i, \leqslant_i)$ in* **PTop**.

PROOF. Let $(s_i)_{i\in I}$ be the family of natural injections, $s_j:X_j\to\amalg_{i\in I}X_i$ and $s_j(x)=(x, j)$. we show that $(\amalg_{i\in I}X_i, \leqslant)$ has the final **PTop**-structure with respect to $(s_i)_{i\in I}$. Let $Z\in$ **PTop** and $g:(\amalg_{i\in I}X_i, \leqslant)\to Z$ be a map. Suppose $g\circ s_i$ is continuous and isotone for each $i\in I$. Since $\amalg_{i\in I}X_i$ has the final topology with respect to $(s_i)_{i\in I}$ g is continuous. Let $(x, i)\leqslant(y, j)$ in $(\amalg_{i\in I}X_i, \leqslant)$. Then $i=j$ and $x\leqslant_i y$. Since $g\circ s_i$ is isotone, $g(x, i)=g\circ s_i(x)\leqslant g\circ s_i(y)=g(y, i)=g(y, j)$.

REMARK. Having had [4] a characterization of the epireflective subcategories of a category **K**, whenever **K** is complete, locally small and colocally small, we are interested in these last two properties for **PTop**.

LEMMA 4. *The category* **PTop** *is locally and colocally small.*

PROOF. It is sufficient to show that the monomorphisms in **PTop** are injective (1 : 1) and the epimorphisms surjective (onto), because then given $m:X\to Y$ a monomorphism, we can always induce an isomorphic copy of X on a subset of Y (and there is only a set of such spaces). Similarly for epimorphisms.

Let $m: X \to Y$ be a monomorphism, a, b in X and $m(a)=m(b)$. We define g, h: $X \to X$ by $g(x)=a$ and $h(x)=b$ for all $x \in X$. Since g and h are continuous isotone maps, $m \circ g = m \circ h$, and m is a monomorphism, $g=h$ and therefore $a=b$.

Let $e: X \to Y$ be an epimorphism, and R the equivalence relation defined on $Y \amalg Y$ by:

$$R=\{((e(x),1),\ (e(x),2)) \mid x \in X\} \cup \{((e(x),2),\ (e(x),1)) \mid x \in X\} \cup \Delta_{Y \amalg Y}$$

Define on $(Y \amalg Y)/R$: $(a,i)_R \leqslant_R (b,j)_R$ if and only if $(a,i) \leqslant (b,j)$, or there exists $c \in X$ such that $a \leqslant e(c)$ and $e(c) \leqslant b$. It is easy to check that $\leqslant_R$ is a well defined partial order and $\nu_R: Y \amalg Y \to (Y \amalg Y)/R$ is continuous and isotone. If we denote by σ_1 and σ_2 the canonical injections $Y \to Y \amalg Y$ we have:

$$\nu_R \circ \sigma_1 \circ e(x) = (e(x),1)_R = (e(x),2)_R = \nu_R \circ \sigma_2 \circ e(x) \quad \text{for all } x \in X.$$

It follows that $\nu_R \circ \sigma_1 \circ e = \nu_R \circ \sigma_2 \circ e$, and, since e is an epimorphism, $\nu_R \circ \sigma_1 = \nu_R \circ \sigma_2$. This means that for an arbitrary $y \in Y$, we have $(y,1)_R = (y,2)_R$ and by the definition of R, $y \in \operatorname{Im} e$.

REMARK. In accordance with Ward Jr. [10], we shall say that if $(x, \leqslant) \in$ **PTop**, the partial order $\leqslant$ is *semicontinuous* if whenever $a \not\leqslant b$ in X there exist two open neighbourhoods U of a and V of b such that $u \not\leqslant b$ and $a \not\leqslant v$ for all $u \in U$ and $v \in V$.

We shall denote by **POTS** the class of **PTop** spaces whose partial order is semicontinuous, and by **KPOTS** the intersection of **KPTop** with **POTS**.

The partial order $\leqslant$ shall be called *continuous* if whenever $a \not\leqslant b$ in X, there exist two open neighbourhoods U of a and V of b such that $u \not\leqslant v$ for all $u \in U$ and $v \in V$.

Every space in **POTS** having continuous partial order is already Hausdorff [10]. We denote by **HOTS** the the class of **POTS** spaces whose partial order is continuous.

THEOREM 1. *The categories* **HPTop**, **HPOTS** *and* **HOTS** *are epireflective in* **PTop**.

PROOF. By Lemmas 2 and 4 it suffices to show [4] that these subcategories of **PTop** are strongly closed with respect to products and equalizers.

From the structure of the products in **PTop** given in Lemma 1, since **H** is closed under products in **Top**, so is **HPTop** under products in **PTop**. Given a family $(X_s)_{s \in S}$ in **HPOTS** (**HOTS**) it is not difficult to see that the partial order of $\Pi_{s \in S} X_s \in$ **PTop** is semicontinuous (continuous).

Consider $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in **PTop** with $X \in$ **HPTop** (or **HPOTS**, or **HOTS**). Since $K = \{x \in X \mid f(x)=g(x)\} \subset X$ and X is Hausdorff, $K \in$ **HPTop**. If $X \in$ **HPOTS** (or **HOTS**), since the semicontinuity (continuity) of the partial order is obviously a hereditary property, we conclude that $k \in$ **HPOTS** (or **HOTS**), and the proof is complete.

THEOREM 2. *If* **K** *is a subcategory of* **H**, *the following statements are equivalent*:

1) **K** *is epireflective in* **H**
2) **KPTop** *is epireflective in* **HPTop**
3) **KPOTS** *is epireflective in* **HPOTS**
4) **KOTS** *is epireflective in* **HOTS.**

PROOF. Let **K** be epireflective in **H**. Since **K** is productive and closed hereditary **KPTop** is complete. One shows as in Theorem 1 that **KPTop** is strongly closed with respect to products and equalizers in **PTop**. Therefore **KPTop** is epireflective in **PTop** and consequently in **HPTop** also. Since we know from Theorem 1 that **HPOTS** is epireflective in **PTop**, the intersection **KPOTS** of **KPTop** and **HPOTS** is epireflective in **HPTop** and so also in **HPOTS**. Similarly, if **KPTop** is epireflective in **HPTop**, one concludes that **KOTS** is epireflective in **HOTS.**

Suppose **KPOTS** is epireflective in **HPOTS**. Let $(X_i)_{i \in I}$ be a family in **K**. By considering $UX_i = (X_i, d)$ where d is the discrete partial order, since **KPOTS** is productive, $\Pi_{i \in I} UX_i \in$ **KPOTS.** Since $\Pi_{i \in I} UX_i$ has $\Pi_{i \in I} X_i$ as its underlying topological space, $\Pi_{i \in I} X_i \in$ **K.** Let $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ be given in **H** with $X \in$ **K**, we consider E the equalizer of $(X, d) \underset{g}{\overset{f}{\rightrightarrows}} (Y, d)$ in **HPOTS**, where $(X, d) \in$ **KPOTS.** Since E inherits the discrete order, its underlying topological space together with the inclusion into X is an equalizer of $X \underset{g}{\overset{f}{\rightrightarrows}} Y$ in **K.** By [4], **K** is epireflective in **H.** Similarly, 4) implies 1).

2. Complete regularization.

DEFINITION 1. Let $E \in$ **PTop.** We call a space $X \in$ **PTop** *E-regular* if there exists a set S and a **PTop**-embedding $e: X \to E^S$. We call an E-regular space *E-compact* if eX is closed in E^S. We denote by **ErOT** and **EcOT** the classes of E-regular and E-compact spaces of **PTop**, respectively.

The following definition is due to Nachbin.

DEFINITION 2. For $X \in$ **PTop**, X is called a *completely regular ordered space* if:
1) if $a \in X$ and if V designates a neighbourhood of a, there exist two continuous realvalued functions f and g on X, where f is increasing (isotone) and g decreasing such that

$0 \leqslant f \leqslant 1, \quad 0 \leqslant g \leqslant 1$

$f(a)=1, \quad g(a)=1$

$\inf[f(x), \quad g(x)]=0$ if $x \in X \backslash V$

2) if $a, b \in X$ and "$a \leqslant b$" is false, there exists a continuous increasing, realvalued function f on X such that $f(a)>f(b)$.

We denote by **CrOR** the category of completely regular ordered spaces as given in the above definition.

LEMMA 5. **IrOT⊂CrOR⊂CrOTS**

PROOF. Let $X \in$ **IrOT**, and S be a set such that X is a subspace of I^S. By [7], Theorem 7 p. 55, we have $X \in$ **CrOR**. If $X \in$ **CrOR** it follows by [7] Prop. 8 p. 59 and [10] Lemma 1 p. 145, that X has a continuous partial order, and by [7] Prop. 6 p. 53, that the underlying topological space of X is completely regular.

THEOREM 3. *Let X be an space in* **PTop**. *X is I-regular if and only if the evaluation map $j: X \rightarrow I^{I_1 X}$ given by $j(x)(f)=f(x)$ is an embedding.*

PROOF. Let X be I-regular. Let $e: X \rightarrow I^S$ be an embedding and $x \neq y$ two elements of X. Since e is an embedding, $e(x) \neq e(y)$ and $p_s \circ e(x) \neq p_s \circ e(y)$ for some $s \in S$. Since $p_s \circ e \in I_1 X$ we obtain $j(x)(p_s \circ e) \neq j(y)(p_s \circ e)$ and $j(x) \neq j(y)$. Therefore j is an injective map. To see that j is a continuous and isotone map, we remark that $I^{I_1 X}$ has the product structure and for every $f \in I_1 X$ the f-projection of j is f, a continuous and isotone map. Suppose now that $j(x) \leqslant j(y)$. Then for every $s \in S$, $e(x)(s)=(p_s \circ e)(x)=j(x)(p_s \circ e) \leqslant j(y)(p_s \circ e)=e(y)(s)$, and we obtain $e(x) \leqslant e(y)$. Since e is an embedding, $x \leqslant y$.

To complete the proof that j is an embedding it is not difficult to show that j is open in its image.

COROLLARY 1. *Let X be a space in* **PTop**. *X is R-regular if and only if the evaluation map $\rho: X \rightarrow R^{C_1 X}$ given by $\rho(x)(f)=f(x)$ is an embedding.*

PROOF. As Theorem 3.

THEOREM 4. *Every completely regular topological space (with the discrete order)* is *I-regular.*

PROOF. Let X be a completely regular space. As is well known, the evaluation $j: X \rightarrow I^{C(X, I)}$ is a **Top**-embedding, and since the order in X is assumed to be discrete, $I_1 X=C(X, I)$ (the set of all continuous maps from X into I). Therefore in order to obtain $X \cong jX$ in **PTop** all we need to show is that the order in jX is discrete. This follows easily from the fact that if $x \neq y$ in X there exists $f \in C(X, I)$ such that $f(x)=0$, $f(y)=1$ and $1-f \in C(X, I)$.

THEOREM 5. **IrOT**=**RrOT**.

PROOF. Since for an arbitrary set S, $I^S \subset R^S$, we see that **IrOT** ⊂**RrOT.** To show the converse we first prove that $R \in$ **IrOT** and, in particular, $R \cong]0,1[$.

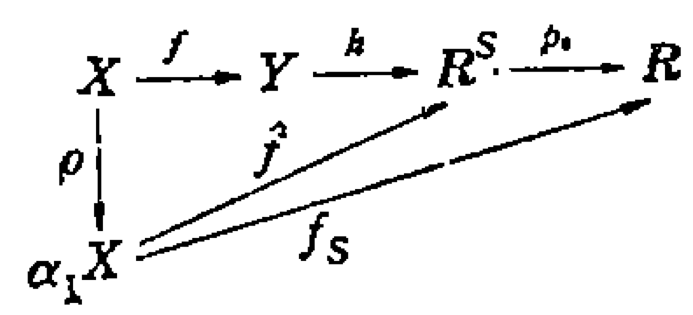
Define $f: R \to]0,1[$ by $f(r)=\frac{1}{2}+\frac{r}{2(r+1)}$. It is very easy to see that f is continuous, isotone, and injective. Moreover, the function $g:]0,1[\to R$ given by $g(x)=\frac{2x-1}{2-2x}$ for $\frac{1}{2} \leqslant x < 1$ and $g(x)=\frac{2x-1}{2x}$ for $0 < x < \frac{1}{2}$ is also continuous isotone and is the inverse of f. Having shown that $R \cong]0,1[$, let $X \in$ **RrOT** and $X \cong Y \subset R^S$. Then $X \cong Y \subset R^S \cong]0, 1[^S \subset I^S$ and it follows that $X \in$ **IrOT.**

REMARK. We denote **IrOT** by **CrORR.** It is clear that **CrORR** is the class of uniformizable (partially) ordered spaces as defined in [7]. From lemma 5 we see that **CrORR** ⊂ **CrOR**, but we have not been able so far to obtain whether these two categories are actually the same.

THEOREM 6. **CrORR** *is an epireflective subcategory of* **PTop.**

PROOF. Given $X \in$ **PTop**, the evaluation map $\rho: X \to R^{C_1X}$ is clearly continuous and isotone although it may not be injective. We call $\alpha_1 X$ the image of ρ. In order to show that α_1:**PTop**→**CrORR** is the desired epireflector, we first show that for every $f \in C_1X$ a unique $\bar{f}: \alpha_1 X \to R$ exists such that $\bar{f} \circ \rho = f$. If p_f is the f-projection we just set $\bar{f}= p_f \circ i$, where i is the inclusion of $\alpha_1 X$ in R^{C_1X}. $\bar{f}$ is clearly continuous and isotone and $\bar{f} \circ \rho = f$. Moreover, since ρ is surjective on $\alpha_1 X$, $\bar{f}$ is unique with respect to these properties.

Consider now the more general case where $f: X \to Y$ is an arbitrary **PTop**-morphism such that $Y \in$ **CrORR**, and let $h: Y \to R^S$ be an embedding. From the result just proved we find for every $s \in S$ a unique continuous isotone map f_S such that $f_S \circ \rho = p_S \circ h \circ f$. From the Universal Property of the Product R^S, there exists a unique continuous isotone map $\hat{f}$ such that for every $s \in S$, $p_S \circ \hat{f} = f_S$.



Having obtained $p_s \circ \hat{f} \circ \rho = p_s \circ h \circ f$ for all $s \in S$, we have $\hat{f} \circ \rho = h \circ f$, and accordingly, $\hat{f} \alpha_1 X = f \circ \rho X = h \circ f X \subset \mathrm{Im}\ h$. Let $a \in \alpha_1 X$ and $\hat{f}(a) = h(y)$. Since h is an embedding, y is unique and we can define a map $\bar{f}: \alpha X \to Y$ by $\bar{f}(a) = y$. Moreover, $\hat{f}(\rho(x)) = h(f(x))$ and therefore $\bar{f}(\rho(x)) = f(x)$. We have obtained $\bar{f}$ which is clearly continuous and isotone; $\bar{f} \circ \rho = f$ and since ρ is surjective, $\bar{f}$ is unique with these properties. This completes the proof.

REMARK. We could have given a parallel proof using the evaluation $j: X \to I^{I_1 X}$ instead of ρ. Suppose we had defined $\alpha_2 X = jX$. Since $(\rho, \alpha_1 X)$ and $(j, \alpha_2 X)$ would then be solutions to the same Universal Problem represented by the Reflexion Property we would have obtained $\alpha_1 X \cong \alpha_2 X$.

3. Compactification and Realcompactification.

If $X \in$ **CrORR**, we denote by $\beta_1 X$ the closure in $I^{I_1 X}$ of jX and by $\nu_1 X$ the closure in R^{CX} of X. Then $X \cong jX$ and jX is a dense subspace of $\beta_1 X$, and, similarly, $X \cong \rho X$ and ρX is a dense subspace of $\nu_1 X$.

THEOREM 7. *The categories of I-compact and R-compact spaces are epireflective in* **CrORR**. *The epireflectors are* β_1 and ν_1 *respectively.*

PROOF. We consider ν_1 since the proof for β_1 is similar and easier. By the same method as in Theorem 6 we obtain that for every $f \in C_1 X$, there exists a unique $\bar{f}: \nu_1 X \to R$ such that $\bar{f} \circ \rho = f$. The uniqueness is assured because ρ is dense in $\nu_1 X$. As in Theorem 6 we consider now the (corresponding) case where $f: X \to Y$ is a **CrORR**-morphism and $Y \in$ **RcOT**. We introduce the embedding $h: Y \to R^S$ with hY closed in R^S and obtain an f_S in **RcOT** such that $f_s \circ \rho = p_s \circ h \circ f$, for every $s \in S$. The rest of the proof is even more similar to the proof of Theorem 6, and so is the proof of the statement concerning β.

THEOREM 8. **IcOT = COTS** $\subset$ **RcOT**

PROOF. It is obvious that **IcOT** $\subset$ **COTS**. Let $X \in$ **COTS**. Since we know that the evaluation map $j: X \to I^{I_1 X}$ is continuous and isotone, we start to show that it is injective. By [7] Theorem 4 p. 48 we know that X is a normally ordered space. Let x, y be two distinct points of X, and without loss of generality, let $x \not\leq y$. Since the partial order on X is continuous, the sets $Ly = \{z \in X \mid z \leq y\}$ and $Mx =$

$\{z\in X\,|\,x\leqslant z\}$ are disjoint. It is not difficult to show that both sets are closed. We conclude from [7] that a continuous isotone function $f\in C_1X$ exists such that $\mathrm{Im}\, f\subset \boldsymbol{I}$, $f(y)=0$ and $f(x)=1$. We call f^* the function in I_1X which has the same graph as f, and since $j(x)(f^*)=f(x)=1\neq 0=f(y)=j(y)(f^*)$, $j(x)\neq j(y)$.

Considering that X is compact, I^{I_1X} Hausdorff, and j continuous, we obtain that j is a closed map. All that remains to the proof is showing that if $j(x)\leqslant j(y)$, then $x\leqslant y$. Suppose it were not so, i.e. say $x\not\leqslant y$. As in our considerations to establish that j is injective, there exists an $f\in I_1X$ such that $j(y)(f)=f(y)=0<1=f(x)=j(x)(f)$, a contradiction.

We have shown that j is an embedding and jX is closed. Therefore $X\in$**IcOT** and **IcOT**$=$**COTS**.

By the same method we show that the evaluation $\rho: X\to \boldsymbol{R}^{C_1X}$ is an embedding and X closed in $\boldsymbol{R}^{C_1X}$. Accordingly, **COTS** $\subset$ **RcOT**.

COROLLARY 2. *Let Y be a space in* **PTop**. *Then Y is in* **CrORR** *if and only if Y is a subspace of some space X in* **COTS**.

PROOF. If Y is in **CrORR**, $Y\subset I^S$ for some set S. Conversely, let $Y\subset X$ and $X\in$**COTS**. By Theorem 8 X is in **IcOT** and therefore in **IrOT**. Since **IrOT** is hereditary, $Y\in$**IrOT**$=$**CrORR**.

COROLLARY 3. *A space X in* **CrORR** *is in* **COTS** *if and only if $\beta_1X\cong X$.*

PROOF. Let $X\in$**COTS**. By the Universal Property shown for β_1 in Theorem 7, there exists a unique map $\beta_1X\xrightarrow{g}X$ such that $g\circ j=1_X$, and 1_{β_1X} is the unique map such that $1_{\beta_1X}\circ j=j$. Since $(j\circ g)\circ j=j\circ(g\circ j)=j\circ 1_X=j$, giving $j\circ g=1_{\beta_1X}$, $\beta_1X\cong X$ follows. The rest of the proof is now obvious.

COROLLARY 4. *All realcompact spaces (with discrete partial order) belong to* **RcOT**.

PROOF. Let X be realcompact with discrete order. Then CX coincides with C_1X, $\rho: X\to \boldsymbol{R}^{CX}$ is a **Top**-embedding with $\rho X=\nu X$ and one shows as in Theorem 4 that the partial order in ρX is discrete. Since $X\cong\nu X$ is a **CrORR**-isomorphism and $\nu X=\nu_1X$, we obtain $X\in$**RcOT**.

4. Connections between β and β_1, ν and ν_1.

Let F be the forgetful functor **HOTS**$\rightarrow$**H** such that $F(X, \leqslant)=X$. If $(X, \leqslant)\in$**COTS**, $X\in$**C**, and therefore $\beta X\cong X$. On the other hand, $\beta_1(X, \leqslant)\cong(X, \leqslant)$, showing that $F\beta_1(X, \leqslant)=\beta X$. Similarly, *if* $(X, \leqslant)\in$**RcOT** $F\nu_1(X, \leqslant)\cong X$. It is natural to wonder whether these results can be generalized for all $X\in$**CrORR**. In this section we generalize the results for discretely ordered **CrORR** spaces and decide the problem in the negative by proving for well-ordered discrete spaces X with cardinality at least that of the continuous that $\beta_1(X, \leqslant)$ is strictly smaller than βX.

THEOREM 9. *For every* $(X, \leqslant)$ *in* **CrORR**, $\beta_1(X, d)\cong(\beta X, d)$ *and there exists a perfect, isotone, surjective map* $p:(\beta X, d)\rightarrow\beta_1(X, \leqslant)$.

PROOF. Let $(X, \leqslant)\in$**CrORR**. By Theorem 4 $(X, d)\in$**CrORR**. Since X is completely regular, for every continuous, isotone $f:(X, d)\rightarrow(Y, \leqslant)$ with $(Y, \leqslant)\in$**COTS**, there exists a unique continuous, isotone map $\bar{f}$ such that if $j_X:(X, d)\rightarrow(\beta X, d)$ is the evaluation map, $\bar{f}\circ j_X=f$.

This shows that $(\beta X, d)$ and $\beta_1(X, d)$ are both solutions to the same Universal Problem and therefore $(\beta X, d)\cong\beta(X, d)$.

Let $h:(X, d)\rightarrow(X, \leqslant)$ be defined by $h(x)=x$ for all $x\in X$. We show that $\beta_1 h$ is a perfect, isotone, and surjective map. Since β_1:**CrORR**$\rightarrow$**COTS** is a functor, $\beta_1 h$ is a **COTS**-morphism, and therefore perfect and isotone. Since $j_{X<}(X, \leqslant)=j_{X<}\circ h(X, d)=\beta_1 h\circ j_{Xd}(X, d)\subset \mathrm{Im}\ \beta_1 h$, we obtain $\beta_1(X, \leqslant)=\Gamma j_{X<}(X, <)\subset\Gamma\ \mathrm{Im}\ \beta_1 h=\mathrm{Im}\ \beta h$. We set p for the composition of $\beta_1 h$ and the isomorphism $(\beta X, d)\cong\beta_1(X, d)$.

COROLLARY 5. *For every* $(X, \leqslant)$ *in* **CrORR**, $\nu_1(X, d)\cong(\nu X, d)$ *and there exists a continuous, isotone and dense map* $q:(\nu X, d)\rightarrow\nu_1(X, \leqslant)$.

PROOF. As in the above Theorem we obtain $\nu_1(X, d)\cong(\nu X, d)$ and $\rho_{X<}(X, \leqslant)\subset\mathrm{Im}\ \nu_1 h\subset\nu_1(X, \leqslant)$.

REMARK. Our next Theorem will show, in particular, that there are spaces X in **CrORR** for which $F\beta(X, \leqslant)\ncong\beta X$.

THEOREM 10. *Let* $(X, \leqslant)$ *be a well ordered, discrete topological space. Then* $(X, \leqslant)\in$**CrORR**. *Moreover, if the cardinality X of X is at least that of the conti-*

nuous, we have: $\overline{\overline{\beta_1(X,\leqslant)}}\leqslant 2^{\overline{\overline{X}}} < 2^{(2^{\overline{\overline{X}}})} = \overline{\overline{\beta X}}$.

PROOF. Since the evaluation $(X,\leqslant)\to I^{I_1(X,\leqslant)}$ is clearly an embedding, $(X,\leqslant)\in$ **CrORR.** To see that $\overline{\overline{\beta_1(X,\leqslant)}}\leqslant 2^{\overline{\overline{X}}}$, we shall first show the existence of an injective map $m: I_1(X,\leqslant)\to(X\times I)^N$. Let X_1 be the set of countable subsets of $X\times I$, $f\in I_1(X,\leqslant)$ and $T_f=\{a\in X \mid f(u)<f(a)$ for all $u<a\}$. We define $t^*=$first $\{a\in T_f \mid t<a\}$ for every $t\in T_f$, and we note that since $t<t^*$ and $t^*\in T_f$, $f(t)<f(t^*)$. Since the set $\{]f(t),f(t^*)[\,\mid t\in T_f\}$ of disjoint intervals of I must be countable, so is T_f. Therefore we can define a map $\gamma: I_1(X,\leqslant)\to X$ by $\gamma(f)=\{(s,f(s))\in \text{graph} f \mid s\in T_f\}$. We shall show that γ is an injective map. Suppose there exist $g,f\in I_1(X,\leqslant)$ such that $g\neq f$ and $\gamma(g)=\gamma(f)$. Let h be such that $g(h)\neq f(h)$. Without loss of generality we may assume that $f(h)<g(h)$. Since $\gamma(g)=\gamma(f)$, we conclude $T_f=T_g$ and $h\notin T_g$. Let $v=$first $\{u\in X \mid g(u)=g(h)\}$. Then $v\leqslant h$, $v\in T_g$ and $g(v)=g(h)$. We obtain $(v,g(v))\in\gamma(g)=\gamma(f)\subset \text{graph } f$, and therefore $g(v)=f(v)$. But this leads to $g(h)=g(v)=f(v)\leqslant f(h)$, which is a contradiction.

Having shown that the map $\gamma: I_1(X,\leqslant)\to X_1$ is injective, we define $\varphi: X_1\to(X\times I)^N$ where $\varphi(\alpha)$ is a counting map of α. Since $\alpha=\text{Im }\varphi(\alpha)$, if $\alpha\neq\beta$ $\text{Im }\varphi(\alpha)\neq\text{Im }\varphi(\beta)$ and $\varphi(\alpha)\neq\varphi(\beta)$. The map $m=\varphi\circ\gamma$ is accordingly injective and $m: I_1(X,\ \leqslant)\to(X\times I)^N$.

We conclude that $\overline{\overline{I_1(X,\leqslant)}}\leqslant\overline{\overline{(X\times I)^N}}=(\overline{\overline{X}}\circ c)^{\aleph_0}=\overline{\overline{X}}^{\aleph_0}=\overline{\overline{X}}$, and, since $\beta_1(X,\leqslant)$ can be embedded into $I^{I_1(X,\leqslant)}$, and $\overline{\overline{I^{I_1(X,\leqslant)}}}\leqslant c^{\overline{\overline{X}}}=2^{\overline{\overline{X}}}$, we obtain $\overline{\overline{\beta_1(X,\leqslant)}}\leqslant 2^{\overline{\overline{X}}}$. On the other hand, we know from [9] that $\overline{\overline{\beta X}}=2^{(2^{\overline{\overline{X}}})}$.

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