Kyungpook Math. J.
Volume 13, Number 1
June, 1973

THE SUM OF TWO RADICAL CLASSES

By Yu-Lee Lee and R. E. Propes

The purpose of this paper is to investigate the concept of the sum of two radical classes.

We shall employ the following notation throughout.

$I \leq R$ denotes I is an ideal of the ring R.

$I \lneqq R$ denotes $I \leq R$ but $I \neq R$.

$R \approx R'$ denotes the rings R and R' are isomorphic.

O, depending upon the context in which it appears denotes the ring O, the ideal O, or the class $\{O\}$.

$L(M)$ denotes the lower radical class determined by the class M of rings.

$U(M)$ denotes the upper radical class determined by the class M of rings.

We shall use the following two equivalent characterizations of radical classes.

THEOREM A [5]. *A subclass P of a universal class W of rings is a radical class if and only if P satisfies the following three conditions.*

(1 *a*) *P is homomorphically closed.*

(2 *a*) *Each ring $R \in W$ has a largest P-ideal, $P(R)$.*

(3 *a*) *If $R \in W$, then $R/P(R)$ is a P-semi-simple ring.*

THEOREM B [1]. *A subclass P of a universal class W of rings is a radical class if and only if P satisfies the following three conditions.*

(1 *b*) *P is homomorphically closed.*

(2 *b*) *If $\{I_\alpha : \alpha \in \Gamma\}$ is a chain of P-ideals of a ring $R \in W$, then $\bigcup_{\alpha \in \Gamma} I_\alpha$ is a P-ideal of R.*

(3 *b*) *If $R \in W$ and if $I \leq R$ such that $I \in P$ and $R/I \in P$, then $R \in P$.*

In what follows, W will denote a universal class of rings or alternative narings, and, unless otherwise specified, $\mathfrak{A}$ and $\mathcal{T}$ will denote radical classes in W.

THEOREM C [2]. *Let P be a radical class in W. If $R \in W$ and $I \leq R$, then $P(I) \leq R$.*

Let M be a subclass of W, and let $H(M)$ be the homomorphic closure of M

in W. For each ring $R\in W$, let $D_1(R)$ be the set of all ideals of R. By induction, define $D_{n+1}(R)$ to be the family of all rings in W which are ideals of some ring in $D_n(R)$. Set $D(R)=\cup\{D_n(R) : n=1, 2, 3, \cdots\}$.

THEOREM D [6]. *$L(M)=\{R\in W : D(R/I)\cap H(M)\neq O$ for each $I \subsetneq R\}$.*

THEOREM E [4]. *If M is a hereditary subclass of W, then $L(M)$ is hereditary.*

THEOREM F [2]. *Let P be a radical class in W. Then P is hereditary if and only if $P(I)=I\cap P(R)$ for each $R\in W$ and each ideal I of R.*

DEFINITION. $\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}=\{R\in W : \mathcal{O}\!\!\!\mathcal{l}(R)+\mathcal{T}(R)=R\}$. We write $(\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T})(R)=\mathcal{O}\!\!\!\mathcal{l}(R)+\mathcal{T}(R)$ for $R\in W$.

PROPOSITION 1. $\mathcal{O}\!\!\!\mathcal{l}\cup\mathcal{T}\subset\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$.

PROOF. Let R be a ring and $R\in\mathcal{O}\!\!\!\mathcal{l}\cup\mathcal{T}$. Then without loss of generality let $R\in\mathcal{O}\!\!\!\mathcal{l}$. Then $\mathcal{O}\!\!\!\mathcal{l}(R)=R$ so that $R=\mathcal{O}\!\!\!\mathcal{l}(R)+\mathcal{T}(R)$ and hence $R\in\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$.

REMARK. $O\in\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$.

PROPOSITION 2. $\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}\subset L(\mathcal{O}\!\!\!\mathcal{l}\cup\mathcal{T})$.

PROOF. Let R be a ring with $R\in\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$. By way of contradiction assume that $R\notin L(\mathcal{O}\!\!\!\mathcal{l}\cup\mathcal{T})$. Then by Theorem D there exists an ideal I of R with $I\neq R$ such that $D(R/I)\cap(\mathcal{O}\!\!\!\mathcal{l}\cup\mathcal{T})=O$. Hence $D(R/I)\cap\mathcal{O}\!\!\!\mathcal{l}=O$ and $D(R/I)\cap\mathcal{T}=O$ and so R/I is both $\mathcal{O}\!\!\!\mathcal{l}$-semi-simple and $\mathcal{T}$-semi-simple. Whence $\mathcal{O}\!\!\!\mathcal{l}(R)\subset I$ and $\mathcal{T}(R)\subset I$. Then $\mathcal{O}\!\!\!\mathcal{l}(R)+\mathcal{T}(R)\subset I$. But $R\in\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$ so that $R\subset I$, and hence $R=I$. This is a contradiction. Thus $R\in\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$ implies $R\in L(\mathcal{O}\!\!\!\mathcal{l}\cup\mathcal{T})$.

REMARK. Since $L(\mathcal{O}\!\!\!\mathcal{l}\cup\mathcal{T})$ is the smallest radical containing both $\mathcal{O}\!\!\!\mathcal{l}$ and $\mathcal{T}$, it follows from Proposition 2 that $\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$ is a radical class if and only if $\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}=L(\mathcal{O}\!\!\!\mathcal{l}\cup\mathcal{T})$.

PROPOSITION 3. *The class $\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$ is homomorphically closed.*

PROOF. Let $R\in\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$ and let R/I be a homomorphic image of R, where $I\leq R$. Let $J/I=\mathcal{O}\!\!\!\mathcal{l}(R/I)$ and let $K/I=\mathcal{T}(R/I)$. Then $(R/I)/(R/\mathcal{O}\!\!\!\mathcal{l}(R/I))=(R/I)/(J/I)\approx R/J$ is $\mathcal{O}\!\!\!\mathcal{l}$-semi-simple. Likewise R/K is $\mathcal{T}$-semi-simple. Thus $\mathcal{O}\!\!\!\mathcal{l}(R)\subset J$ and $\mathcal{T}(R)\subset K$ and so $R=\mathcal{O}\!\!\!\mathcal{l}(R)+\mathcal{T}(R)\subset J+K$, i.e., $R=J+K$. Then $R/I=(J+K)/I=J/I+K/I=\mathcal{O}\!\!\!\mathcal{l}(R/I)+\mathcal{T}(R/I)$ and so $R/I\in\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$.

DEFINITION. An ideal I of a ring R is called an *$(\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T})$-ideal* if $I\in\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T}$.

PROPOSITION 4. *$\mathcal{O}\!\!\!\mathcal{l}(R)+\mathcal{T}(R)$ is the largest $(\mathcal{O}\!\!\!\mathcal{l}+\mathcal{T})$-ideal of the ring R.*

PROOF. First we show that $\mathfrak{A}(R)+\mathcal{T}(R)$ is an $(\mathfrak{A}+\mathcal{T})$-ideal of the ring R. Plainly $\mathfrak{A}(R)+\mathcal{T}(R)$ is an ideal of R, because both $\mathfrak{A}$ and $\mathcal{T}$ are radicals. Clearly $\mathfrak{A}(R)\subset \mathfrak{A}(\mathfrak{A}(R)+\mathcal{T}(R))\leq \mathfrak{A}(R)+\mathcal{T}(R)$ and $\mathcal{T}(R)\subset \mathcal{T}(\mathfrak{A}(R)+\mathcal{T}(R))\leq \mathfrak{A}(R)+\mathcal{T}(R)$. Hence $\mathfrak{A}(R)+\mathcal{T}(R)=\mathfrak{A}(\mathfrak{A}(R)+\mathcal{T}(R))+\mathcal{T}(\mathfrak{A}(R)+\mathcal{T}(R))$ and so $\mathfrak{A}(R)+\mathcal{T}(R)\in\mathfrak{A}+\mathcal{T}$.

To see that $\mathfrak{A}(R)+\mathcal{T}(R)$ is the largest $(\mathfrak{A}+\mathcal{T})$-ideal of R, let $I\leq R$ and $I\in\mathfrak{A}+\mathcal{T}$. Then $I=\mathfrak{A}(I)+\mathcal{T}(I)$. But by Theorem C, $\mathfrak{A}(I)\subset\mathfrak{A}(R)$ and $\mathcal{T}(I)\subset\mathcal{T}(R)$. Therefore $I=\mathfrak{A}(I)+\mathcal{T}(I)\subset\mathfrak{A}(R)+\mathcal{T}(R)$.

THEOREM 2. *If $S(\mathfrak{A})\cap\mathcal{T}=0$, $S(\mathcal{T})\cap\mathfrak{A}=0$, and $\mathfrak{A}\cap\mathcal{T}=0$, then $\mathfrak{A}+\mathcal{T}$ is a radical class. Recall that $S(\mathfrak{A})$ and $S(\mathcal{T})$ are the semi-simple classes of the radicals $\mathfrak{A}$* and *$\mathcal{T}$ respectively.*

PROOF. In view of propositions 3 and 4, it only remains to prove that $R/(\mathfrak{A}(R)+\mathcal{T}(R))$ is $(\mathfrak{A}+\mathcal{T})$-semi-simple for an arbitrary ring R. For this let $\mathfrak{A}(R/\mathfrak{A}(R)+\mathcal{T}(R))=J/(\mathfrak{A}(R)+\mathcal{T}(R))$ and $\mathcal{T}(R/(\mathfrak{A}(R)+B(R)))=K/(\mathfrak{A}(R)+\mathcal{T}(R))$. Then R/J is $\mathfrak{A}$-semi-simple and R/K is $\mathcal{T}$-semi-simple. Now $J/K\cap J\approx(K+J)/K\leq R/K$, and R/K is $\mathcal{T}$-semi-simple. Therefore $J/J\cap K\in S(\mathcal{T})$. Similarly $K/J\cap K\in S(\mathfrak{A})$. But $\mathfrak{A}(R)+\mathcal{T}(R)\subset J\cap K$. Thus $J/(\mathfrak{A}(R)+\mathcal{T}(R))$ can be mapped homomorphically onto $J/J\cap K$, and $K/(\mathfrak{A}(R)+\mathcal{T}(R))$ can be mapped homomorphically onto $K/J\cap K$. But $J/(\mathfrak{A}(R)+\mathcal{T}(R))\in\mathfrak{A}$ and $K(\mathfrak{A}(R)+\mathcal{T}(R))\in\mathcal{T}$. Hence $J/J\cap K\in\mathfrak{A}\cap S(\mathcal{T})=0$ and $K/J\cap K\in\mathcal{T}\cap S(\mathfrak{A})=0$. Whence $J=J\cap K$ and $K=J\cap K$ and so $J=K=J\cap K$. It follows that $(\mathfrak{A}+\mathcal{T})(R/(\mathfrak{A}(R)+\mathcal{T}(R)))=0$.

THEOREM 2. *Assume that $S(\mathcal{T})\cap\mathfrak{A}=0$ and $S(\mathfrak{A})\cap\mathcal{T}=0$ and that $(I, M\leq R$, $I/M\in\mathfrak{A}\cap\mathcal{T}$, $M\supset\mathfrak{A}(R))$ implies $I\in\mathfrak{A}\cap\mathcal{T}$. Then $\mathfrak{A}+\mathcal{T}$ is a radical class.*

PROOF. Just as in the case of Theorem 1 it remains to prove that $R/(\mathfrak{A}(R)+\mathcal{T}(R))$ is $(\mathfrak{A}+\mathcal{T})$-semi-simple. For this let $\mathfrak{A}(R/(\mathfrak{A}(R)+\mathcal{T}(R))=J/(\mathfrak{A}(R)+\mathcal{T}(R))$ and let $\mathcal{T}(R/(\mathfrak{A}(R)+\mathcal{T}(R))=K/(\mathfrak{A}(\mathrm{R})+\mathcal{T}(R))$. Now $R/J\in S(\mathfrak{A})$ and $R/K\in S(\mathcal{T})$. Therefore $K/K\cap J\approx(K+J)/J$ is $\mathfrak{A}$-semi-simple and $J/K\cap J\approx(K+J)/K$ is $\mathcal{T}$-semi-simple. Now $\mathfrak{A}(R)+\mathcal{T}(R)\subset J\cap K$ so that $K/(\mathfrak{A}(R)+\mathcal{T}(R))$ can be mapped homomorphically onto $K/J\cap K$ and $J/(\mathfrak{A}(R)+\mathcal{T}(R))$ can be mapped homomorphically onto $J/J\cap K$. Thus $K/J\cap K\in S(\mathfrak{A})\cap\mathcal{T}=0$ and $J/J\cap K\in S(\mathcal{T})\cap\mathfrak{A}=0$. Hence $K\subset J\cap K$ and $J\subset J\cap K$ and so $J=K$. Then $J\cap K/(\mathfrak{A}(R)+\mathcal{T}(R))\in\mathfrak{A}\cap\mathcal{T}$ and so by the condition of our theorem $J\cap K\in\mathfrak{A}\cap\mathcal{T}$. Thus $J\in\mathfrak{A}\cap\mathcal{T}\subset\mathfrak{A}$ and $K\in\mathfrak{A}\cap\mathcal{T}\subset\mathcal{T}$ and so $J\subset\mathfrak{A}(R)$ and $K\subset\mathcal{T}(R)$. Therefore $(J+K)/(\mathfrak{A}(R)+\mathcal{T}(R))=0$, i.e., $R/(\mathfrak{A}(R)+\mathcal{T}(R))$ is $(\mathfrak{A}+\mathcal{T})$-semi-simple.

THEOREM 3. *If* $S(\mathscr{T})\cap\mathfrak{A}=O$ *and* $S(\mathfrak{A})\cap\mathscr{T}=O$ *and* $\mathfrak{A}+\mathscr{T}=\mathfrak{A}\cup\mathscr{T}$, *then* $\mathfrak{A}+\mathscr{T}$ *is a radical class.*

PROOF. We show that $R/(\mathfrak{A}(R)+\mathscr{T}(R))$ is $(\mathfrak{A}+\mathscr{T})$-semi-simple. Let $\mathfrak{A}(R/(\mathfrak{A}(R)+\mathscr{T}(R)))=J/(\mathfrak{A}(R)+\mathscr{T}(R))$ and let $\mathscr{T}(R/(\mathfrak{A}(R)+\mathscr{T}(R)))=K/(\mathfrak{A}(R)+\mathscr{T}(R))$. Now $\mathfrak{A}(R)+\mathscr{T}(R)\in\mathfrak{A}+\mathscr{T}=\mathfrak{A}\cup\mathscr{T}$. Say $\mathfrak{A}(R)+\mathscr{T}(R)\in\mathfrak{A}$. Then by condition (3b) we have $J\in\mathfrak{A}$ and so $J\subset\mathfrak{A}(R)\subset\mathfrak{A}(R)+\mathscr{T}(R)$. Hence $J=\mathfrak{A}(R)+\mathscr{T}(R)$. Now $R/J\in S(\mathfrak{A})$ and $K/(\mathfrak{A}(R)+\mathscr{T}(R))\in\mathscr{T}$, while $K/J\cap K\approx(J+K)/J\leq R/J$ and $K/J\cap K$ is a homomorphic image of $K/(\mathfrak{A}(R)+\mathscr{T}(R))$. Thus $K/J\cap K\in S(\mathfrak{A})\cap\mathscr{T}=O$. Therefore $K\subset J\cap K\subset J=\mathfrak{A}(R)+\mathscr{T}(R)$. Hence $R/(\mathfrak{A}(R)+\mathscr{T}(R))$ is $\mathfrak{A}+\mathscr{T}$-semi-simple. We arrive at the same conclusion if $\mathfrak{A}(R)+\mathscr{T}(R)\in\mathscr{T}$.

THEOREM 4. $\mathfrak{A}+\mathscr{T}$ *is a radical class if and only if* $R/I\in\mathfrak{A}+\mathscr{T}$ *and* $I\in\mathfrak{A}+\mathscr{T}$ *implies* $R\in\mathfrak{A}+\mathscr{T}$.

PROOF. If $\mathfrak{A}+\mathscr{T}$ is a radical class, then clearly the condition must be satisfied, because the condition is condition (3b) of Theorem B. Thus assume that the condition holds. To prove that $\mathfrak{A}+\mathscr{T}$ is a radical class it suffices to show that $R/(\mathfrak{A}(R)+\mathscr{T}(R))$ is $(\mathfrak{A}+\mathscr{T})$-semi-simple. Hence let $K/(\mathfrak{A}(R)+\mathscr{T}(R))=\mathscr{T}(R/(\mathfrak{A}(R)+\mathscr{T}(R)))$ and let $J/(\mathfrak{A}(R)+\mathscr{T}(R))=\mathfrak{A}(R/(\mathfrak{A}(R)+\mathscr{T}(R)))$. Now $J/(\mathfrak{A}(R)+\mathscr{T}(R))\in\mathfrak{A}\subset\mathfrak{A}+\mathscr{T}$ and $K/(\mathfrak{A}(R)+\mathscr{T}(R))\in\mathscr{T}\subset\mathfrak{A}+\mathscr{T}$, and $\mathfrak{A}(R)+\mathscr{T}(R)\in\mathfrak{A}+\mathscr{T}$. Therefore, by the condition, $J,K\in\mathfrak{A}+\mathscr{T}$. But then $J=\mathfrak{A}(J)+\mathscr{T}(J)\subset\mathfrak{A}(R)+\mathscr{T}(R)$ and $K=\mathfrak{A}(K)+\mathscr{T}(K)\subset\mathfrak{A}(R)+\mathscr{T}(R)$. Thus $J+K\subset\mathfrak{A}(R)+\mathscr{T}(R)$ and so $O=(J+K)/(\mathfrak{A}(R)+\mathscr{T}(R))=J/(\mathfrak{A}(R)+\mathscr{T}(R))+K/(\mathfrak{A}(R)+\mathscr{T}(R))=\mathfrak{A}(R/(\mathfrak{A}(R)+\mathscr{T}(R)))+\mathscr{T}(R/(\mathfrak{A}(R)+\mathscr{T}(R)))=(\mathfrak{A}+\mathscr{T})(R/(\mathfrak{A}(R)+\mathscr{T}(R)))$.

Next we give an example of radical classes $\mathfrak{A}$ and $\mathscr{T}$ for which $\mathfrak{A}+\mathscr{T}$ is not a radical class.

EXAMPLE. Let Z denote the ordinary ring of integers and let $R=Z/(4)=\{0+(4), 2+(4), 3+(4)\}$. Let $A=\{0+(4), 2+(4)\}$ and $B=R/A$. Set $\mathfrak{A}=L(H(\{A\}))$ and $\mathscr{T}=L(H(\{B\}))$. Then $R\notin\mathscr{T}$, because $D(R)\cap H(\{B\})=O$; and $A\notin\mathscr{T}$, because $D(A)\cap H(\{B\})=O$. Therefore $\mathscr{T}(R)=O$. Also, $R\notin\mathfrak{A}$, because $D(R/A)\cap H(\{A\})=O$. But $A\in\mathfrak{A}$, clearly. Hence $(\mathfrak{A}+\mathscr{T})(R)=\mathfrak{A}(R)+\mathscr{T}(R)=\mathfrak{A}(R)+O=A$. Thus $R/(\mathfrak{A}(R)+\mathscr{T}(R))=R/A=B\neq O$, and so $O\neq R/(\mathfrak{A}(R)+\mathscr{T}(R))\in\mathscr{T}\subset\mathfrak{A}+\mathscr{T}$. This shows that R is not $(\mathfrak{A}+\mathscr{T})$-semi-simple and hence that $\mathfrak{A}+\mathscr{T}$ is not a radical class.

DEFINITION. $S(\mathfrak{A}+\mathscr{T})=\{R\in W:(\mathfrak{A}+\mathscr{T})(R)=O\}$.

We have seen that in general $\mathcal{A}+\mathcal{T}$ is not a radical class, however, we are able to prove $S(\mathcal{A}+\mathcal{T})$ is a semi-simple class.

DEFINITION [5. P. 17]. A subclass Q of W is a *semi-simple class* if Q has the following properties.

(1s) If $R\in Q$ and $I\leq R$, then I has no non-zero homomorphic image in Q.

(2s) If $R\in W$ and $R\notin Q$, then R has a non-zero ideal $I\in\{A\in W: A$ has no non-zero homomorphic image in Q$\}$.

LEMMA. *$S(\mathcal{A}+\mathcal{T})$ is hereditary.*

PROOF. Let $R\in S(\mathcal{A}+\mathcal{T})$ and let $I\leq R$. Now $\mathcal{A}(R)+\mathcal{T}(R)=O$ and $\mathcal{A}(I)\subset\mathcal{A}(R)$ and $\mathcal{T}(I)\subset\mathcal{T}(R)$ so that $\mathcal{A}(I)+\mathcal{T}(I)=O$. Thus $I\in S(\mathcal{A}+\mathcal{T})$.

THEOREM 5. *$S(\mathcal{A}+\mathcal{T})$ is a semi-simple class.*

PROOF. We must show that $S(\mathcal{A}+\mathcal{T})$ satisfies conditions (1s) and (2s). By the above Lemma $S(\mathcal{A}+\mathcal{T})$ satisfies condition (1s). Thus let $R\notin S(\mathcal{A}+\mathcal{T})$. Then $\mathcal{A}(R)+\mathcal{T}(R)\neq O$ and $\mathcal{A}(R)+\mathcal{T}(R)\leq R$. Since $\mathcal{A}(R)+\mathcal{T}R)\in\mathcal{A}+\mathcal{T}$, then every non-zero homomorphic image of $\mathcal{A}(R)+\mathcal{T}(R)$ is in $\mathcal{A}+\mathcal{T}$ and hence not in $S(\mathcal{A}+\mathcal{T})$. Hence condition (2s) is satisfied and $S(\mathcal{A}+\mathcal{T})$ is a semi-simple class.

REMARK. In fact, $S(\mathcal{A}+\mathcal{T})=S(\mathcal{A})\cap S(\mathcal{T})$.

By the Lemma we have from [3, Theorem 2] that $U(S(\mathcal{A}+\mathcal{T}))=\{R\in W$: every non-zero homomorphic image $f(R)\notin S(\mathcal{A}+\mathcal{T})\}$. We shall show that $L(\mathcal{A}\cup\mathcal{T})=U(S(\mathcal{A}+\mathcal{T}))$.

THEOREM 6. $S(L(\mathcal{A}\cup\mathcal{T}))=S(\mathcal{A}+\mathcal{T})$.

PROOF. Let $R\in S(\mathcal{A}+\mathcal{T})=S(\mathcal{A})\cap S(\mathcal{T})$ and let $I=L(\mathcal{A}\cup\mathcal{T})(R)$. Now $I\leq R\in S(\mathcal{A})\cap S(\mathcal{T})$, thus $I\in S(\mathcal{A})\cap S(\mathcal{T})$, because semi-simple classes are hereditary. Hence $D(I)\cap\mathcal{A}=O=D(I)\cap\mathcal{T}$. Therefore $D(I)\cap(\mathcal{A}\cup\mathcal{T})=O$ and so $I=O$, since $I\in L(\mathcal{A}\cup\mathcal{T})$. Thus $S(\mathcal{A}+\mathcal{T})\subset S(L(\mathcal{A}\cup\mathcal{T}))$. Now let $R\in S(L(\mathcal{A}\cup\mathcal{T}))$. Then $L(\mathcal{A}\cup\mathcal{T})(R)=O$. If $R\notin S(\mathcal{A})$, then there exists a non-zero ideal I of R such that $I\in\mathcal{A}\subset L(\mathcal{A}\cup\mathcal{T})$, which is a contradiction. Similarly, we reach a contradiction if $R\notin S(\mathcal{T})$. So we must have $R\in S(\mathcal{A})\cap S(\mathcal{T})=S(\mathcal{A}+\mathcal{T})=S(\mathcal{A}+\mathcal{T})$. Therefore $S(L(\mathcal{A}\cup\mathcal{T}))\subset S(\mathcal{A}+\mathcal{T})$.

COROLLARY. $L(\mathcal{A}\cup\mathcal{T})=U(S(\mathcal{A}+\mathcal{T}))$.

PROOF. By Theorem 6 $S(\mathcal{A}+\mathcal{T})=S(L(\mathcal{A}\cup\mathcal{T}))$. Therefore $U(S(\mathcal{A}+\mathcal{T}))=U(S(L(\mathcal{A}\cup\mathcal{T})))=L(\mathcal{A}\cup\mathcal{T})$.

REMARK. $\mathcal{A}+\mathcal{T}$ is a radical class if and only if $R/(\mathcal{A}(R)+\mathcal{T}(R))\in S(\mathcal{A})\cap S(\mathcal{T})$ for each ring R.

PROPOSITION 5. *If each $\mathcal{A}$ and $\mathcal{T}$ is hereditary and if $\mathcal{A}+\mathcal{T}$ is a radical, then $\mathcal{A}+\mathcal{T}$ is a hereditary radical.*

PROOF. If $\mathcal{A}+\mathcal{T}$ is a radical, then $\mathcal{A}+\mathcal{T}=L(\mathcal{A}\cup\mathcal{T})$. Since each of $\mathcal{A}$ and $\mathcal{T}$ is hereditary, the $\mathcal{A}\cup\mathcal{T}$ is a hereditary class. Then by Theorem E $L(\mathcal{A}\cup\mathcal{T})$ is hereditary.

PROPOSITION 6. *If $\mathcal{A}+\mathcal{T}$ is a hereditary class, then $I\cap\mathcal{A}(R)+I\cap\mathcal{T}(R)=I\cap(\mathcal{A}(R)+\mathcal{T}(R))$ for each ring R and each ideal I of R.*

PROOF. Let R be a ring and let $I\leq R$. Then $\mathcal{A}(I)+\mathcal{T}(I)\subset I\cap\mathcal{A}(R)+I\cap\mathcal{T}(R)\subset I\cap(\mathcal{A}(R)+\mathcal{T}(R))\subset I$. But $I\cap(\mathcal{A}(R)+\mathcal{T}(R))\leq\mathcal{A}(R)+\mathcal{T}(R)\in\mathcal{A}+\mathcal{T}$. Hence $I\cap(\mathcal{A}(R)+\mathcal{T}(R))\in\mathcal{A}+\mathcal{T}$, i.e., $I\cap(\mathcal{A}(R)+\mathcal{T}(R))$ is an $(\mathcal{A}+\mathcal{T})$-ideal of I. Since $\mathcal{A}(I)+\mathcal{T}(I)$ is the largest $(\mathcal{A}+\mathcal{T})$-ideal of I, then $I\cap(\mathcal{A}(R)+\mathcal{T}(R))\subset\mathcal{A}(I)+\mathcal{T}(I)$.

THEOREM 7. *Let $\mathcal{A}$ and $\mathcal{T}$ be hereditary radicals. Then the class $\mathcal{A}+\mathcal{T}$ is hereditary if and only if $I\cap\mathcal{A}(R)+I\cap\mathcal{T}(R)=I\cap(\mathcal{A}(R)+\mathcal{T}(R))$ for each ring R and each ideal I of R.*

PROOF. If $\mathcal{A}+\mathcal{T}$ is hereditary, the condition follows from Proposition 6. Thus suppose the condition holds and let $R\in\mathcal{A}+\mathcal{T}$ and $I\leq R$. Since each of $\mathcal{A}$ and $\mathcal{T}$ is a hereditary radical, we have by Theorem F, $\mathcal{A}(I)+\mathcal{T}(I)=I\cap\mathcal{A}(R)+I\cap\mathcal{T}(R)$. By the condition we have $I\cap\mathcal{A}(R)+I\cap(\mathcal{A}(R)+\mathcal{T}(R))=I\cap R=I$. Thus $\mathcal{A}(I)+\mathcal{T}(I)=I$ and so $I\in\mathcal{A}+\mathcal{T}$.

Kansas State University and
the University of Wisconsin-Milwaukee

REFERENCES

[1] S.A. Amitsur, A general theory of radicals, II. *Radicals in rings and bicategories*, Am. J, Math. 76 (1954), 100—125.

[2] T. Anderson, N. Divinsky, A. Sulinski, *Hereditary radicals in associative and alternative rings*, Can. Jour. Math. 17 (1965), 594—603.

[3] N.J. Divinsky, *Rings and Radicals*, University of Toronto Press, Toronto, (1965).

[4] A.E. Hoffman, W.G. Leavitt, *Properties inherited by the lower radical*, (to appear).

[5] A.G. Kurosh, *Radicals of rings and algebras* (Russian), Mat. Sb., 33 (75) (1953), 13—26.

[6] Yu-Lee Lee, *On the construction of lower radical properties of rings*, Pacific J. of Math., Vol. 28, No. 2, (1969).