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ON THE K-PROXIMITIES

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The theory of proximity spaces was essentially discovered in the early 1950's by Efremovič when he axiomatically characterized the proximity relation "A is near B", which is denoted by $A\ \delta\ B$, for subsets A and B of a set X.

Efremovič's axioms of proximity relation δ are as follows:

P1. $A \delta B$ implies $B \delta A$

P2. $(A \cup B) \delta C$ if and only if $A \delta C$ or $B \delta C$

P3. $A \delta B$ implies $A \neq \phi$, $B \neq \phi$

P4. $A \bar{\delta} B$ implies there exists a subset E such that $A \bar{\delta} E$ and $(X-E) \bar{\delta} B$

P5. $A \cap B \neq \phi$ implies $A \delta B$.

A binary relation δ satisfing axioms P1-P5 on the power set of X is called the *E-proximity* (Efremovič's proximity) on X. Defining the closure of a subset A of a proximity space (X, δ) to be the set $\{x \in X \mid x \delta A\}$, Efremovic showed that a topology $\mathscr{T}(\delta)$ can be introduced in X and that this induced topology is completely regular. He also showed that every completely regular space $(X, \mathscr{T})$ admits a compatible proximity δ on X such that $\mathscr{T}(\delta) = \mathscr{T}$.

In this work we propose some generalization of the concept of the Efremovič's proximity, which we call a "K-proximity" and examine some of its properties. We also try to characterize the topological structure based on this K-proximity.

K-proximity and E-proximity.

DEFINITION 1. Let δ be a binary relation between X and $\mathscr{P}(X)$ such that

i) $x \delta A \cup B$ iff $x \delta A$ or $x \delta B$

ii) $x \bar{\delta} \phi$ for all $x \in X$

iii) $x \in A$ implies $x \delta A$

iv) $x \bar{\delta} A$ implies there is a subset E such that $x \bar{\delta} E$ and $y \bar{\delta} A$ for all $y \in X-E$.

The binary relation $\delta \subset X \times \mathscr{P}(X)$ is called the *K-proximity* on X iff δ satisfies the axioms i)—iv). The pair (X, δ) is called the *K-proximity space.*

One can easily show that the E-proximity on X implies the K-proximity on X.

THEOREM 1. *Every E-proximity on X is also an K-proximity on X.*

PROOF. P1 and P2 implies i), P3 implies ii), and P5 implies iii). If $A=\{x\}$ and $A \not\delta B$ then from P4 there exists a subset $E \subset X$ with $x \not\delta E$ and $X-E \not\delta B$. Hence for each $y \in X-E$ we have $y \not\delta B$. This means that P1 and P4 imply iv).

We now give an example of an K-proximity which is not an E-proximity. Let $S=\{0, 1\}$ be the Sierpinski space with the topology $\mathscr{T}=\{\phi, \{0\}, S\}$ then by the following theorem 2 there is a K-proximity δ on S with $\mathscr{T}(\delta)=\mathscr{T}$. But since S is not completely regular, S can not admits an E-proximity.

Now we shall introduce the Efremovič proximity δ_1 from the K-proximity δ replacing the axiom iv) in K-proximity by the stronger one.

DEFINITION 2. A binary relation δ defined between X and $\mathscr{P}(X)$ is called the *E-proximity* on X iff δ satisfies the axioms i), ii), iii) in the definition 1, and

iv′) For each subset $E \subset X$ there is a point $x \in X$ such that either $x \delta A$, $x \delta E$ or $x \delta B$, $x \delta X\text{-}E$, then we have $x \delta A$ and $x \delta B$.

DEFINITION 3. In a K-proximity space (X, δ) let δ_1 be a binary relation on $\mathscr{P}(X)$ defined as follows:
For each subsets A, B of X, $A \delta_1 B$ iff there is a point $x \in X$ such that $x \delta A$, $x \delta B$.

THEOREM 2. *The binary relation δ_1 on $\mathscr{P}(X)$ defined in definition 3 is the Efremovič proximity on X.*

PROOF. 1) It is clear that $A \delta_1 B$ implies $B \delta_1 A$.

2) $(A \cup B) \delta_1 C \Leftrightarrow$ there is a point x such that $x \delta A \cup B$ and $x \delta C$

$\Leftrightarrow (x \delta A$ or $x \delta B)$ and $x \delta C$

$\Leftrightarrow (x \delta A, x \delta C)$ or $(x \delta B, x \delta C)$

$\Leftrightarrow A \delta_1 C$ or $B \delta_1 C$.

3) $A \delta_1 B \Rightarrow$ there is a point x such that $x \delta A$, $x \delta B$

$\Rightarrow A \neq \phi$ and $B \neq \phi$.

4) Suppose that for each subset $E \subset X$, $A \delta_1 E$ or $B \delta_1 X\text{-}E$. Hence for some point $x \in X$ we have either $x \delta A$, $x \delta E$ or $x \delta B$, $x \delta X\text{-}E$, therfore by iv′) $x \delta A$ and $x \delta B$, that is, $A \delta_1 B$.

5) $A \cap B \neq \phi \Rightarrow \exists x \in X$ with $x \in A \cap B \Rightarrow x \in A$ and $x \in B \Rightarrow x \delta A$ and $x \delta B \Rightarrow A \delta_1$ B.

In what follows we introduce some properties of the K-proximity.

LEMMA 1. *If $x \delta A$ and $A \subset B$ then $x \delta B$.*

PROOF. By i) $x \delta A \Rightarrow x \delta A \cup B \Rightarrow x \delta B$.

THEOREM 3. *In the K-proximity space* (X, δ) *if* A^{δ} *is defined to be a set* $\{x \mid x\,\delta\,A,\ x \in X\}$ *for each subset* $A \subset X$, *then* δ *is a Kuratowski's closure operator. Hence we can introduce the topology* $\mathscr{T}(\delta)$ on X *by* δ *and for each subset* $A \subset X$

$$\overline{A} = A^{\delta}.$$

where $\overline{A}$ *is the closure of* A *with respect to* $\mathscr{T}(\delta)$.

PROOF. Now we show that δ is a closure operator.

1) For each $x \in X$, $x \not\delta \phi$. Hence $x \notin \phi^{\delta}$, that is $\phi^{\delta} = \phi$.

2) If $x \in A$ then $x\,\delta\,A$ or $x \in A^{\delta}$. Therefore $A \subset A^{\delta}$.

3) Let $x \not\delta$ A then by iv) there exists a set $E \subset X$ such that $x \not\delta E$ and $y \not\delta A$ for all $y \in X\text{-}E$. If $z \in A^{\delta}$, then $z\,\delta\,A$. Hence $z \in E$, that is $A^{\delta} \subset E$. Since $x \not\delta E$ we have $x \not\delta A^{\delta}$. This means that $x \in A^{\delta\delta}$ implies $x \in A^{\delta}$ or $A^{\delta\delta} \subset A^{\delta}$. Therefore $A^{\delta\delta} = A^{\delta}$.

4) $x \in (A \cup B)^{\delta} \Leftrightarrow x\delta(A \cup B) \Leftrightarrow x\,\delta\,A$ or $x\,\delta\,B \Leftrightarrow x \in A^{\delta}$ or $x \in B^{\delta} \Leftrightarrow x \in A^{\delta} \cup B^{\delta}$ That is, $(A \cup B)^{\delta} = A^{\delta} \cup B^{\delta}$.

THEOREM 4. *Let* $(X, \mathscr{T})$ *be a topological space. If a binary relation* $\delta \subset X \times \mathscr{P}(X)$ *is defined by* $x\,\delta\,A$ *iff* $x \in \overline{A}$, *then* δ *is a K-proximity on* X *and the topology* $\mathscr{T}(\delta)$ *induced by* δ *is the given topology* $\mathscr{T}$.

PROOF. 1) $\overline{\phi} = \phi \Rightarrow x \not\delta \phi$ for all $x \in X$.

2) $x\delta(A \cup B) \Leftrightarrow x \in \overline{A \cup B} = \overline{A} \cup \overline{B} \Leftrightarrow x \in \overline{A}$ or $x \in \overline{B} \Leftrightarrow x\,\delta\,A$ or $x\,\delta\,B$.

3) $x \in A \Rightarrow x \in \overline{A} \Rightarrow x\,\delta\,A$.

4) $x \not\delta A \Rightarrow x \notin \overline{A} \Rightarrow x \notin (\overline{\overline{A}}) \Rightarrow x \not\delta \overline{A}$.

Now let $\overline{A} = E$ then $x \not\delta E$ and $y \not\delta A$ for all $y \in X\text{-}E = X\text{-}\overline{A}$. Since $x \in \overline{A} \Leftrightarrow x\delta A \Leftrightarrow x \in A^{\delta}$, we have $\overline{A} = A^{\delta}$, that is $\mathscr{T}(\delta) = \mathscr{T}$.

THEOREM 5. *The topological space* X *is* T_1 *iff there is K-proximity* δ *on* X *satisfing the following condition*:

v) $x\delta\{y\} \Rightarrow x = y$.

PROOF. X is T_1. $\Rightarrow$ There is a binary relation δ on X satisfing conditions i)–iv). Then $x \in A^{\delta} \Leftrightarrow x\delta A$. Hence $x\delta\{y\} \Rightarrow x \in \{y\}^{\delta} = \{y\}$ since X is T_1. That is $x = y$. Conversely if $x\delta\{y\}$ implies that $x = y$, then $\{y\}^{\delta} = \{y\}$, that is, X is T_1.

LEMMA 2. $x\delta\{y\}$, $y\delta A \Rightarrow x\delta A$.

PROOF. $x \not\delta A \Rightarrow$ There is a subset $E \subset X$ such that $x \not\delta E$, $z \not\delta A$ for all $z \in X - E \Rightarrow y \notin E$ (if $y \in E$ then $x\delta\{y\}$, $y \in E$ so we have $x\delta E) \Rightarrow y \in X - E$, that is $y \not\delta A$. It is a contradiction.

It is easy to show that the following theorem is true in the K-proximity space (x, δ).

THEOREM 6. 1) *A subset G of X is open iff $x\bar{\delta}(X-G)$ for every x in G.*
2) *If $x \bar{\delta} A$, then $x \in \text{Int}(X-A)$.*

DEFINITION 4. Let δ_1, δ_2 be K-proximities on X. We define

$$\delta_1 < \delta_2 \text{ iff } x\delta_2 A \Rightarrow x\delta_1 A$$

The above is expressed by saying that δ_2 is *finer* than δ_1, or δ_1 is *coarser* than δ_2.

The following theorem shows that a finer K-proximity structure induces a finer topology.

THEOREM 7. *Let δ_1, δ_2 be two K-proximities defined on a set X. Then we have:*
1) *$\delta_1 < \delta_2$ implies $\mathscr{T}(\delta_1) \subset \mathscr{T}(\delta_2)$.*

2) *Let $\mathscr{T}_1$ and $\mathscr{T}_2$ be two topologies on X, and δ_1 and δ_2 be the K-proximities on X defined as in the theorem 4 with respect to $\mathscr{T}_1$ and $\mathscr{T}_2$ respectively. Then $\mathscr{T}_1 \subset \mathscr{T}_2$ implies $\delta_1 < \delta_2$.*

PROOF. 1) $G \in \mathscr{T}(\delta_1) \Rightarrow x \in G$ implies that $x\bar{\delta}_1(X-G)$
$\Rightarrow x \in G$ implies that $x\bar{\delta}_2(X-G)$
$\Rightarrow G \in \mathscr{T}_2(\delta_2)$.

2) $x\delta_2 A \Rightarrow x \in A^{\delta_2} \Rightarrow x \in A^{\delta_1}$ since $\mathscr{T}_1 \subset \mathscr{T}_2$ implies $A^{\delta_2} \subset A^{\delta_1} \Rightarrow x\delta_1 A$.

Alternate description of K-proximity.

Given a proximity space (X, δ), a subset B may said to be a proximity neighborhood of A iff $A\bar{\delta}(X-B)$. An analogous concept, that of a K-proximity neighbourhood, can be introduced in a K-proximity space and furnishes an alternative approach to the study of K-proximity spaces.

DEFINITION 5. A subset A of a K-proximity space (X, δ) is a *δ-neighbourhood* of a point x in X (in symbols $x \ll A$) iff $x\bar{\delta}(X-A)$.

LEMMA 2. *Let (X, δ) be a K-proximity space and let* Int$A$ *denote the interior of A. Then $x \ll A$ implies $x \ll \text{Int}A$. Therefore $x \in \text{Int}A$, showing that a δ-neighbourhood is a topological neighbourhood.*

PROOF. $x \ll A \Rightarrow x\bar{\delta}(X-A) \Rightarrow x\bar{\delta}(X-A)^{\delta} = X - \text{Int}A \Rightarrow x \ll \text{Int}A$.

LEMMA. 3. *If $x \bar{\delta} A$ then there is a subset B of X such that $x \ll B$ and $y \bar{\delta} A$ for* *every y in B.*

PROOF. If $x\bar{\delta}A$ then by axiom iv) there is a subset E of X such that $x\bar{\delta}E$ and $y\bar{\delta}A$ for every y in $X-E$. Let $X-E=B$ then $x\ll B$ and $y\bar{\delta}A$ for every y in B.

THEOREM 8. *Given a K-proximity space (X, δ), the relation $\ll$ satisfies the following properties:*

1) *$x\ll X$ for every x in X.*

2) *$x\ll A$ implies $x\in A$.*

3) *If $x \ll A$ and $A\subset B$ then $x\ll B$.*

4) *$x\ll A$ implies there is a subset B of X such that $x\ll B$ and $y\ll A$ for every y in B.*

5) *if $x\ll A_i$ for $i=1, 2, \cdots, n$ then $x\ll \bigcap_{i=1}^{n} A_i$.*

6) *If δ is T_1, then $x\ll(X-\{y\})$ iff $x \neq y$.*

PROOF. 1) Since $x\bar{\delta}\phi$, $x\ll X$.

2) is clear.

3) $x\ll A,\ A\subset B \Rightarrow x\bar{\delta}(X-A),\ X-B\subset X-A \Rightarrow x\bar{\delta}(X-B) \Rightarrow x\ll B$.

4) $x\ll A \Rightarrow x\bar{\delta}(X-A) \Rightarrow$ By lemma 3 there is a B such that $x\ll B$ and $y\ll A$ for every y in B.

5) $x\bar{\delta}(X-A_i)$ for $i=1, 2, \cdots, n \Rightarrow x\bar{\delta} \bigcup_{i=1}^{n} (X-A_i) \Rightarrow x\bar{\delta}(X-\bigcap_{i=1}^{n} A_i)$

$\Rightarrow x\ll \bigcap_{i=1}^{n} A_i$.

6) $x\ll(X-\{y\}) \Leftrightarrow x\bar{\delta}\{y\} \Leftrightarrow x\neq y$.

THEOREM 9. *If $\ll$ is binary relation between X and $\mathscr{P}(X)$ satisfying the properties 1)—5) in the theorem 8 and δ is defined by*

$$x\bar{\delta}A \textit{ iff } x\ll(X-A),$$

then δ is a K-proximity on X. A is a δ-neighbourhood of x iff $x\ll A$.

PROOF. i) $x\bar{\delta}A,\ x\bar{\delta}B \Leftrightarrow x\ll(X-A),\ x\ll(X-B) \Leftrightarrow x\ll(X-A)\cap(X-B) \Leftrightarrow x\bar{\delta}(A\cup B)$.

ii) $x\in X \Rightarrow x\ll X \Rightarrow x\bar{\delta}\phi$.

iii) $x\bar{\delta}A \Rightarrow x\ll(X-A) \Rightarrow x\in(X-A) \Rightarrow x\notin A$.

iv) If $x\bar{\delta}A$ then $x\ll(X-A)$. By 4) there is a B such that $x\ll B$ and $y\ll(X-A)$ for every y in B. Hence let $E=X-B$ then $x\bar{\delta}E$ and $y\bar{\delta}A$ for every y in B. That is, there is a subset E of X such that $x\bar{\delta}E$ and $y\bar{\delta}A$ for every y in $X-E$.

It is easily seen that if $\ll$ satisfies the additional property 6) in the theorem 8 then δ is T_1.

K-proximity mappings and subspaces.

In the study of general topological spaces, continuous functions play an important role. A similar role is played by proximity mappings in proximity spaces. Their analogue in the theory of K-proximity spaces can be introduced as follows.

DEFINITION 6. Let (X, δ_1) and (Y, δ_2) be two K-proximity spaces. A function $f:X\to Y$ is said to be a *K-proximity mapping* iff $x\delta_1 A$ implies $f(x)\delta_2 f(A)$

THEOREM 10. *Let (X, δ_1) and (Y, δ_2) be two K-proximity spaces and let $f: X \to Y$ be a function. The following properties of f are equivalent:*

1) *f is a K-proximity mapping.*

2) *$y\not\delta_2 B$ implies $x\not\delta_1 f^{-1}(B)$ for each $x\in f^{-1}(y)$.*

3) *$y\ll_2 B$ implies $x\ll_1 f^{-1}(B)$ for each $x\in f^{-1}(y)$.*

PROOF. 1) $\Rightarrow$ 2). Suppose that there is some $x\in f^{-1}(y)$ such that $x\ \delta_1 f^{-1}(B)$. Then $f(x)\delta_2 f(f^{-1}(B))$ and $f(f^{-1}(B))\subset B$. Hence $y\ \delta_2 B$.

2) $\Rightarrow$ 3). If $y\ll_2 B$ then $y\not\delta_2(Y-B)$. By 2) we have $x\not\delta_1 f^{-1}(Y-B)$ for every $x\in f^{-1}(y)$. That is, $x\not\delta_1(X-f^{-1}(B))$ for every $x\in f^{-1}(y)$ or $x\ll_1 f^{-1}(B)$ for every $x\in f^{-1}(y)$.

3) $\Rightarrow$ 1). If $f(x)\not\delta_2 f(A)$ then $f(x)\ll_2(Y-f(A))$. By 3) we have $x'\ll_1 f^{-1}(Y-f(A))$ for every $x'\in f^{-1}(f(x))$. Since $x\in f^{-1}(f(x))$, $x\ll_1(X-f^{-1}(f(A))$ and therefore $x\not\delta_1 f^{-1}(f(A))$ or $x\not\delta_1 A$.

It is easy to see that the composition of two K-proximity mappings is a K-proximity mapping. The following theorem is similar to the well-known result: a proximity mapping is continuous with respect to the induced topologies.

THEOREM 11. *A mapping $f: (X, \delta_1)\to(Y, \delta_2)$ is continuous with respect to $\mathscr{T}(\delta_1)$ and $\mathscr{T}(\delta_2)$ iff f is a K-proximity mapping.*

PROOF. If f is continuous and $x\delta_1 A$ then $x\in\overline{A}$. By the continuity of f we have $f(x)\in\overline{f(A)}$, or $f\ (x)\delta_2\ f(A)$. That is, f is a K-proximity mapping.

If f is a K-proximity mapping and $x\in\overline{A}$ then $x\delta_1 A$ and also $f(x)\ \delta_2 f(A)$. That is, $f(x)\in f\overline{(A)}$ or f is continuous.

THEOREM 12. *Given a function $f{:}X\to(Y,\delta_1)$, the binary relation δ defined by $x\ \delta\ A$ iff $f(x)\delta_1 f(A)$, is the coarsest K-proximity on X such that f is a K-proximity mapping.*

PROOF. We first show that δ is a K-proximity on X.

i) $x\delta(A\cup B)\Leftrightarrow f(x)\delta_1\ f(A\cup B)=f(A)\cup f(B)\Leftrightarrow f(x)\delta_1 f(A)$ or $f(x)\delta_1 f(B)$ $\Leftrightarrow x\delta A$ or $x\delta B$.

ii) $x\in X\Rightarrow f(x)\in Y\Rightarrow f(x)\not\delta_1\phi\Rightarrow f(x)\not\delta_1\ f(\phi)\Rightarrow x\not\delta\phi$.

iii) $x\in A\Rightarrow f(x)\in f(A)\Rightarrow f(x)\delta_1 f(A)\Rightarrow x\delta A$.

iv) $x\not\delta A\Rightarrow f(x)\not\delta_1 f(A)\Rightarrow$ there is a subset E_1 such that $f(x)\not\delta_1 E_1$ and $y\not\delta_1 f(A)$ for every y in $Y-E_1\Rightarrow$ Let $E=\mathrm{f}^{-1}(E_1)$ then we have $f(E)\subset E_1$, $f(x)\not\delta_1 f(E)$ and $y\not\delta_1 f(A)$ for every y in $(Y-E_1)\cap f(X)\Rightarrow x\not\delta E$ and $x'\not\delta\ A$ for every x' in $f^{-1}(Y-E_1)\cap f^{-1}(f(X))=(X-f^{-1}(E_1))\cap X=X-E$.

Let δ_0 be any K-proximity on X such that $f{:}(X,\delta_0)\to(Y,\delta_1)$ is a K-proximity mapping, then $x\delta_0 A$ implies $f(x)\delta_1 f(A)$ or $x\delta A$. That is, $\delta<\delta_0$.

DEFINITION 7. Two K-proximity spaces (X,δ_1) and (Y,δ_2) are said to be *K-proximally isomorphic* iff there exists a one-to-one mapping f from X onto Y such that both f and f^{-1} are K-proximally mappings. Such a mapping f is called a *K-proximity isomorphism.*

It follows from the Theorem 11 that two K-proximity spaces are K-proximally isomorphic iff they are homeomorphic.

DEFINITION 8. Let (X,δ) be a K-proximity space, and $Y\subset X$. The induced K-proximity δ_Y on Y is the coarsest K-proximity such that the inclusion mapping $i:Y\to X$ is a K-proximity mapping. The K-proximity space (Y,δ_Y) is called the *subspace* of (X,δ) and δ_Y is called the *induced K-proximity.*

Product spaces and quotient spaces.

We next consider the product of a family $\{(X\alpha,\delta_\alpha):\alpha\in I\}$ of K-proximity spaces. Let $X=\Pi\{X_\alpha:\alpha\in I\}$ denote the Cartesian product of these spaces. We define a product K-proximity $\delta=\Pi\{\delta_\alpha:\alpha\in I\}$ on X as follows:

DEFINITION 9. Let $x\in X$ and A a subset of X. Define $x\delta A$ iff for each finite cover $\mathfrak{A}=\{A_1, A_2, \cdots, A_n\}$ of A there is A_i such that $P_\alpha(x)\delta_\alpha P_\alpha(A_i)$ for each $\alpha\in I$. Where P_α denotes the projection of X onto X_α.

THEOREM 13. *The binary relation δ defined in the definition 9 is a K-proximity on the product set X.*

PROOF. i) Let A and B be subsets of X. If $x\delta A$ and $\mathfrak{A}=\{E_1, E_2, \cdots, E_n\}$ is a finite cover of $A\cup B$ then $\mathfrak{A}$ is also a a cover of A and there is some E_i in $\mathfrak{A}$ such that $P_\alpha(x)\delta_\alpha P_\alpha(E_i)$ for each $\alpha\in I$. That is, $x\delta(A\cup B)$.

Suppose that $x\not\delta A$ and $x\not\delta B$, then there is some finite covers $\mathfrak{A}=\{A_1, A_2, \cdots, A_m\}$ of A and $\mathcal{L}=\{B_1, B_2, \cdots, B_n\}$ of B such that for each $A_i\in\mathfrak{A}$ there is $\alpha_i\in I$ with $P_{\alpha_i}(x)\not\delta_{\alpha_i}P_{\alpha_i}(A_i)$ and for each $B_j\in\mathcal{L}$ there is $\alpha_j\in I$ with $P_{\alpha_j}(x)\not\delta_{\alpha_j}P_{\alpha_j}(B_j)$. $\mathfrak{A}\cup\mathcal{L}=\{A_1, A_2, \cdots, A_m, B_1, B_2, \cdots, B_n\}$ is a cover of $A\cup B$ and there is no member A_i or B_j in $\mathfrak{A}\cup\mathcal{L}$ such that $P_\alpha(x)\ \delta_\alpha P_\alpha(A_i)$for each $\alpha\in I$ or $P_\alpha(x)\ \delta_\alpha P_\alpha(B_j)$ for each $\alpha\in I$. Hence we have $x\not\delta(A\cup B)$.

ii) Since $\mathfrak{A}=\{\phi\}$ is a finite cover of ϕ and $P_\alpha(x)\ \not\delta_\alpha P_\alpha(\phi)$ for each $\alpha\in I$ we have $x\not\delta\phi$.

iii) If $x\in A$ and $A=A_1\cup A_2\cup\cdots\cup A_n$ then there is some A_i such that $x\in A_i$. Hence for each $\alpha\in I$ we have $P_\alpha(x)\ \delta_\alpha P_\alpha(A_i)$, that is, $x\delta A$.

iv) If $x\not\delta A$ then there is some finite cover $\mathfrak{A}=\{A_1, A_2, \cdots, A_n\}$ of A such that for each A_i in $\mathfrak{A}$, $P_{\alpha_i}(x)\ \not\delta_{\alpha_i}P_{\alpha_i}(A_i)$ for some $\alpha_i\in I$.

Since $(X_{\alpha_i}, \delta_{\alpha_i})$ is a K-proximity space for $i=1, 2, \cdots, n$, there is $E_i\subset X_{\alpha_i}$ such that $P_{\alpha_i}(x)\ \not\delta_{\alpha_i}E_i$ and $y_i\ \not\delta_{\alpha_i}\ P_{\alpha_i}(A_i)$ for each $y_i\in X_{\alpha_i}-E_i$. Let $E=P_{\alpha_1}^{-1}(E_1)\cup\cdots\cup P_{\alpha_n}^{-1}(E_n)$ then $x\not\delta E$ since $\{P_{\alpha_i}^{-1}(E_i)\}$ is a cover of E and for each i, $P_{\alpha_i}(x)\ \not\delta_{\alpha_i}P_{\alpha_i}(P_{\alpha_i}^{-1}(E_i))$. On the other hand $y\not\delta A$ for each y in $X-E$. For if $y\in X-E$ then $y\notin P_{\alpha_i}^{-1}(E_i)$ for each i or $P_{\alpha_i}(y)\notin E_i$, hence $P_{\alpha_i}(y)\in X_{\alpha_i}-E_i$ for each i and $P_{\alpha_i}(y)\ \not\delta_{\alpha_i}P_{\alpha_i}(A_i)$ for each i.

DEFINITION 10. Let $\{(X_\alpha, \delta_\alpha)|\alpha\in I\}$ be a family of K-proximity spaces $(X_\alpha, \delta_\alpha)$. The pair (X, δ), where $X=\Pi X_\alpha$, $\delta=\Pi\delta_\alpha$, is called *the product K-proximity space* of the family.

THEOREM 14. A mapping f from a K-proximity space (Y, δ_1) to a product K-proximity space $X=\Pi X_\alpha$ is a K-proximity mapping iff the composition $P_\alpha\circ f: Y$

$\to X_\alpha$ is a K-proximity mapping for each projection P_α.

PROOF. We need only prove that if each $P_\alpha \circ f$ is a K-proximity mapping then so is f. Let $y \in Y$ and $B \subset Y$. And suppose that $y\delta_1 B$ and $f(y) \not\delta f(B)$ then there is some cover $\mathcal{A} = \{A_1, \cdots, A_n\}$ of $f(B)$ such that for each $A_i \in \mathcal{A}$, $P_{\alpha_i}(f(y)) \not\delta_{\alpha_i} P_{\alpha_i}(A_i)$ for some $\alpha_i \in I$. Since $\{f^{-1}(A_1), \cdots, f^{-1}(A_n)\}$ is a cover of B and $y\delta_1 B$, we have $y\delta_1 f^{-1}(A_j)$ for some A_j. Hence $P_\alpha \circ f(y)\ \delta_\alpha P_\alpha \circ f(f^{-1}(A_j))$ for each $\alpha \in I$ since $P_\alpha \circ f$ is a K-proximity mapping. That is, $P_\alpha(f(y)) \delta_\alpha P_\alpha(A_j)$ for each $\alpha \in I$. This is contradict with the fact $P_{\alpha_j}(f(y))\ \not\delta_{\alpha_j} P_{\alpha_j}(A_j)$. Therefore $f(y) \delta f(B)$, that is, f is a K-proximity mapping.

COROLLARY. *The product K-proximity $\delta = \Pi\delta_\alpha$ is the coarsest K-proximity on $X = \Pi X_\alpha$ for which each projection P_α is a K-proximity mapping.*

We now turn our attention to the category of all K-proximity spaces. We first consider a category $\mathcal{A}$ whose class of objects is the class of all K-proximity spaces and for each $(Y, Z) \in \mathcal{A} \times \mathcal{A}$ whose Hom (Y, Z) is the set of all K-proximity mappings of Y into Z and also consider the product K-proximity space $X = \Pi\{X_\alpha | \alpha \in I\}$ of a family $\{(X_\alpha, \delta_\alpha) | \alpha \in I\}$ of K-proximity spaces. Let Z^Y be the set of all K-proximity mappings from Y into Z, where Y and Z are K-proximity spaces, and let $(X)^Y$ be the cartesian product $\Pi\{X_\alpha^Y | \alpha \in I\}$ then we have a category $\mathcal{L}$ of sets whose class of objects is the class $\mathcal{L} = \{(X)^Y | Y \in \mathcal{A}\}$ and whose Hom $((X)^Y, (X)^Z)$ is the set of all functions from $(X)^Y$ into $(X)^Z$. Now let $T: \mathcal{A} \to \mathcal{L}$ be a contravariant functor such that for each $Y \in \mathcal{A}$, $T(Y) = (X)^Y$ and for each $g \in$ Hom (Y, Z) in $\mathcal{A}$, $T(g): (X)^Z \to (X)^Y$ with $T(g)((f_\alpha)) = (g \circ f_\alpha)$ where $(f_\alpha) \in (X)^Z = \Pi\{X_\alpha^Z | \alpha \in I\}$, then T is really a contravariant functor.

Combining the above discussion and the theorem 14 we obtain the following result:

THEOREM 15. *The contravariant functor $T: \mathcal{A} \to \mathcal{L}$ has a universal element $((p_\alpha), X)$, where $P_\alpha: X \to X_\alpha$ is the projection from the product space $X = \Pi\{X_\alpha | \alpha \in I\}$ to X_α and $(p_\alpha) \in (X)^X$.*

In the following we shall introduce the concept of quotient K-proximity.

THEOREM 16. *Let (X, δ) be a K-proximity space and let $f: X \to Y$ be a mapping, where Y is any set. If we define $y\delta_1 B$ iff each f-saturated closed subset of X*

containing $f^{-1}(B)$ *contains* $f^{-1}(y)$, *then* δ_1 *is a K-proximity on Y and f is a K-proximity mapping.* (*or* δ_1 *is the finest K-proximity on Y such that f is a K-proximity mapping.*)

PROOF. We first show that δ_1 is a K-proximity on Y.

i) Suppose that $y\delta_1(A\cup B)$ and $y\not\delta_1 B$ then each f-saturated closed set F containing $f^{-1}(A\cup B)$ contains $f^{-1}(y)$ and there is f-saturated closed set G containing $f^{-1}(B)$ such that $G\cap f^{-1}(y)=\phi$. Consequently each f-saturated closed set H containing $f^{-1}(A)$ contains $f^{-1}(y)$, since if $H\cap f^{-1}(y)=\phi$ then the closed saturated $H\cup G$ containing $f^{-1}(A)\cup f^{-1}(B)$ does not contain $f^{-1}(y)$ and it is a contradiction. Hence $y\delta_1 A$. Suppose that $y\delta_1 A$ then each f-saturated closed set F containing $f^{-1}(A)$ contains $f^{-1}(y)$. Hence each f-saturated closed set H containing $f^{-1}(A)\cup f^{-1}(B)$ also contains $f^{-1}(y)$. That is, $y\delta_1(A\cup B)$.

ii) Since the empty set ϕ is a f-saturated closed set containing $\phi=f^{-1}(\phi)$ such that $f^{-1}(y)\cap\phi=\phi$ for each y in Y, we have $y\not\delta_1\phi$ for each y in Y.

iii) If $y\in A$ then $f^{-1}(y)\subset f^{-1}(A)$ and each f-saturated closed set F containing $f^{-1}(A)$ also contains $f^{-1}(y)$. Therefore we have $y\delta_1 A$.

iv) If $y\not\delta_1 A$ then there is a f-saturated closed set F containing $f^{-1}(A)$ such that $F\cap f^{-1}(y)=\phi$. Let $E=f(F)$ then $f^{-1}(E)=F$ and $y\not\delta_1 E$. If $z\in Y-E$ then $f^{-1}(z)\subset f^{-1}(Y-E)=X-f^{-1}(E)=X-F$. Hence $z\not\delta_1 A$ for each $z\in Y-E$.

Next we show that $f:(X,\delta)\to(Y,\delta_1)$ is a K-proximity mapping. Let $x\delta A$ and let F be a f-saturated closed set containing $f^{-1}(f(A))$ then $x\delta F$ because of $F\supset f^{-1}(f(A))\supset A$. Hence $x\in F$ and $f^{-1}(f(x))\cap F\neq\phi$. Consequently $F\supset f^{-1}(f(x))$ since F is saturated. This means that $f(x)\delta_1 f(A)$.

THEOREM 17. *In the theorem* 16, δ_1 *is the finest K-proximity on Y such that f is a K-proximity mapping.*

PROOF. Let δ_0 be any K-proximity on Y such that f is a K-proximity mapping. And let $y\not\delta_0 B$ then $y\not\delta_0\overline{B}$ and we have $x\not\delta f^{-1}(\overline{B})$ for each x in $f^{-1}(y)$, that is, $f^{-1}(y)\cap f^{-1}(\overline{B})=\phi$. Since $f^{-1}(\overline{B})$ is a f-saturated closed set containing $f^{-1}(B)$, $y\not\delta_1 B$.

DEFINITION 11. Let (X, δ) be a K-proximity space and let $f: X \to Y$ be a mapping. The finest K-proximity δ_1 on Y such that f is a K-proximity mapping is called the *quotient K-proximity* for Y relative to f and the K-proximity δ on X.

THEOREM 18. *Let f be a K-proximity mapping of a space X onto a space Y and let Y have the quotient K-proximity. Then a mapping g on Y to a K-proximity space Z is a K-proximity mapping iff the composition $g \circ f$ is a K-proximity, mapping.*

PROOF. Let $g \circ f$ be a K -proximity mapping and let $g(y) \not\delta_2 g(B)$ then $g(y) \not\delta_2 \overline{g(B)}$. Since $g \circ f$ is a K-proximity mapping, for each x in $f^{-1}g^{-1}(g(y))$, $x \not\delta_x f^{-1}g^{-1}(\overline{g(B)})$ or for each x in $f^{-1}(y)$, $x \not\delta_x f^{-1}g^{-1}(\overline{g(B)}) \supset f^{-1}(B)$ or $x \not\delta_x f^{-1}(B)$ and $f^{-1}\left(g^{-1}(\overline{g(B)})\right)$ is a f-saturated closed set containing $f^{-1}(B)$ in X. Hence $y \not\delta_r B$. The converse is clear.

Let (X, δ) be any given K-proximity space and let R be an equivalence relation on X, then we have a quotient K-proximity δ_R on the quotient set X/R defined by the projection $p_R: X \to X/R$. Now let us consider the category $\mathfrak{A}$ of all K-proximity spaces defined in the theorem 15 and a category $\mathcal{L}$ of sets whose class of objects is $\mathcal{L} = \{Y^X \mid Y \in \mathfrak{A}\}$ and whose Hom (Y^X, Z^X) is the set of all functions from Y^X into Z^X. Then we have a covariant functor $F: \mathfrak{A} \to \mathcal{L}$ such that for each $Y \in \mathfrak{A}$ $F(Y) = Y^X$, and for each $g \in$ Hom (Y, Z), $F(g): Y^X \to Z^X$ with $F(g)(f) = g \circ f$. If we consider a subfunctor H of F such that for each $Y \in \mathfrak{A}$, $H(Y) = \{f \in Y^X \mid (x, y) \in R \Rightarrow f(x) = f(y)\}$ and for each $g \in$ Hom (Y, Z), $H(g)(f) = g \circ f$, then we obtain the following result from the above discussion and the theorem 18:

THEOREM 19. *The subfunctor H of the covariant functor $F: \mathfrak{A} \to \mathcal{L}$ has a universal element $(p_R, X/R)$.*

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