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NOTE ON SUBMANIFOLDS WITH (f, g, u, v, λ)-STRUCTURE IN AN EVEN-DIMENSIONAL EUCLIDEAN SPACE

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§0. Introduction.

Let M be a differentiable manifold of class C^∞. If there exist in M a (1, 1) type tensor field f, two vector fields U, V, two 1-forms u, v, a function λ and a Riemannian metric g satisfying the conditions;

$$f^2X=-X+u(X)U+v(X)V,$$
$$fU=-\lambda V,\quad fV=+\lambda U,$$
$$u(fX)=+\lambda v(X),\quad v(fX)=-\lambda u(X),$$
$$u(U)=v(V)=1-\lambda^2,\quad v(U)=u(V)=0,$$
$$g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y),$$
$$g(U, X)=u(X),\quad g(V, X)=v(X)$$

for any vector fields X and Y, then M is said to have an (f,g,u,v,λ)-structure (cf. [6]).

Submanifolds of codimension 2 in an even-dimensional Euclidean space induce an (f, g, u, v, λ)-structure (cf. [6]).

Recently submanifolds of codimension 2 in an even-dimensional Euclidean space have been studied by S. S. Eum [2], U-Hang Ki [2], [3], [5], Jin Suk Pak [3], M. Okumura [4], [7], K. Yano [6], [7] and many authors.

The main purpose of the present paper is to study complete submanifolds of codimension 2 in an even-dimensional Euclidean space such that $fH-Hf=0$, $\nabla_X\lambda=\phi v(X)$, ϕ being a differentiable function.

In §1, we recall the properties of a submanifold of codimension 2 in an even-dimensional Euclidean space.

In §2, we find several lemmas to be useful in §3.

In §3, we investigate properties of a complete submanifold of codimension 2 in an even-dimensional Euclidean space under our assumptions stated above.

§ 1. Preliminaries.

Let E be a $(2n+2)$-dimensional Euclidean space and X the position vector starting from the origin of E and ending at a point of E.

The E being even-dimensional, it can be regarded as a flat Kählerian manifold with the numerical structure tensor $F: F^2=-I$, where I denotes the unit tensor and $FY\cdot FZ=Y\cdot Z$ for arbitrary vector fields Y and Z, where the dot denotes the inner product of vectors of E.

We consider a $2n$-dimensional orientable manifold M covered by a system of coordinate neighborhoods $\{U: x^h\}$, where here and in the sequel the indices $h, i, j, \cdots$ run over the range $\{1, 2, \cdots, 2n\}$.

We assume that M is immersed in E by $X: M\to E$ and put $X_i=\partial_i X$, $\partial_i=\partial/\partial x_i$. Then X_i are $2n$ linearly independent vector fields tangent to the submanifold M and $g_{ji}=X_j\cdot X_i$ are local components of the tensor representing the Riemannian metric induced on M from that of E.

We denote by C and D two mutually orthogonal unit normals to the submanifold M such that X_i, C, D form the positive orientation of E, then M induces an (f, g, u, v, λ)-structure which satisfies the following;

$$\nabla_j f_i{}^h=-h_{ji}u^h+h_j{}^h u_i-k_{ji}v^h+k_j{}^h v_i, \tag{1.1}$$

$$\nabla_j u_i=-h_{jt}f_i{}^t-\lambda k_{ji}+l_j v_i, \tag{1.2}$$

$$\nabla_j v_i=-k_{jt}f_i{}^t+\lambda h_{ji}-l_j u_i, \tag{1.3}$$

$$\nabla_j\lambda=-h_{jt}v^t+k_{jt}u^t, \tag{1.4}$$

where ∇_j denotes the operator of covariant differentiation with respect to the Riemannian connection, h_{ji} and k_{ji} are components of the second fundamental tensors with respect to C and D respectively defined by $h_j{}^h=h_{ji}g^{ih}$ and $k_j{}^h=k_{ji}g^{ih}$, and l_j are components of the third fundamental tensor, that is, component of the connection induced on the normal bundle (cf. [5], [7]).

In the sequel, we need the structure equations of the submanifold M, i.e., the following equations of Gauss

$$K_{kjih}=h_{ji}h_{kh}-h_{jh}h_{ki}+k_{ji}k_{kh}-k_{jh}k_{ki}, \tag{1.5}$$

where K_{kjih} are covariant components of the curvature tensor of M, and equations of Codazzi and Ricci

$$\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki}=0, \tag{1.6}$$

(1.7) $$\nabla_k k_{ji} - \nabla_j k_{ki} + l_k h_{ji} - l_j h_{ki} = 0,$$

(1.8) $$\nabla_j l_i - \nabla_i l_j + h_{jt} k_i^{\ t} - h_{it} k_j^{\ t} = 0,$$

K. Yano and U-Hang Ki proved in [5]

THEOREM 1.1. *Let M be a complete submanifold of codimension 2 in an even-dimensional Euclidean space E^{2n+2} such that the scalar curvature of M is constant and there are global unit normals C and D to M which are parallel in the normal bundle.*

If $fH = Hf$ and $fK = -Kf$ hold, where H and K are the second fundamental tensors of M respectively with respect to C and D, f being the tensor field of type (1, 1) *appearing in the induced structure (f, g, u, v, λ) of M, then M is in E^{2n+2}, provided that $\lambda(1-\lambda^2)$ is non-zero almost everywhere in M, congruent to one of the following submanifolds:*

$$E^{2n},\ S^{2n}(r),\ S^n(r)\times S^n(r),\ S^l(r)\times E^{2n-l}\ (l=1, 2, \cdots, 2n-1),$$
$$S^k(r)\times S^k(r)\times E^{2n-2k}\ (k=1, 2, \cdots, n-1),$$

where $S^k(r)$ denotes a k-dimensional sphere of radius $r(>0)$ imbedded naturally in E^{2n+2}.

§2. The case in which f and H commute and $\nabla_X \lambda = \phi v(X)$.

We suppose that f and H commute in M, that is,

(2.1) $$f_j^{\ t} h_t^{\ h} - h_j^{\ t} f_t^{\ h} = 0,$$

which is equivalent to

(2.2) $$h_{jt} f_i^{\ t} + h_{it} f_j^{\ t} = 0.$$

Under this conditions K. Yano and U-Hang Ki have proved in [5]

LEMMA 2.1. *Let $X(M)$ be a submanifold of codimension 2 of E such that the global unit normals C and D are parallel in the normal bundle. Assume that* (2.1) *is satisfied and the function $\lambda(1-\lambda^2)$ is non-zero almost everywhere in M.*

Then we have

(2.3) $$h_{jt} u^t = p u_j, \qquad h_{jt} v^t = p v_j,$$

(2.4) $$h_t^{\ h} h_i^{\ t} = p h_i^{\ h}, \qquad p = constant,$$

(2.5) $$\nabla_k h_{ji}=0,$$

p being given by $(1-\lambda^2)p=h_{ts}u^t u^s=h_{ts}v^t v^s$.

We now prove that

LEMMA 2.2. *Let X(M) be a submanifold of codimension 2 of E such that the global unit normals C and D are parallel in the normal bundle and the function* $\lambda(1-\lambda^2)$ *is almost everywhere non-zero. Assume that* (2.1) *and*

(2.6) $$\nabla_j\lambda=\phi v_j,$$

ϕ being non-zero differentiable function on M, are satisfied; then

(2.7) $$k_{jt}u^t=(p+\phi)v_j,$$

(2.8) $$k_{jt}v^t=(p+\phi)u_j+\beta v_j,$$

(2.9) $$(1-\lambda^2)(\nabla_j\phi)=(v^t\nabla_t\phi)v_j+\lambda\phi\beta u_j,$$

(2.10) $$(1-\lambda^2)(k_{jt}f_i{}^t-k_{it}f_j{}^t)=\lambda\beta(u_jv_i-u_iv_j),$$

where β is given by $(1-\lambda^2)\beta=k_{ts}v^tv^s$.

PROOF. From (1.4), (2.3) and (2.6), we have (2.7).

Differentiating (2.6) covariantly and using (2.3) with $l_j=0$, we find

$$\nabla_k\nabla_j\lambda=(\nabla_k\phi)v_j+\phi(-k_{kt}f_j{}^t+\lambda h_{kj}),$$

from which, taking skew-symmetric parts,

(2.11) $$(\nabla_k\phi)v_j=(\nabla_j\phi)v_k+\phi(k_{kt}f_j{}^t-k_{jt}f_k{}^t).$$

Transvecting (2.11) with v^j and substituting (2.7), we find

(2.12) $$(1-\lambda^2)(\nabla_j\phi)=\{v^t\nabla_t\phi+\lambda\phi(p+\phi)\}v_j-\phi k_{ts}v^s f_j{}^t.$$

Substituting (2.12) into (2.11), we get

(2.13) $$k_{ts}v^s(f_j{}^tv_i-f_i{}^tv_j)=(1-\lambda^2)(k_{it}f_j{}^t-k_{jt}f_i{}^t).$$

Transvecting (2.13) with $f_h{}^j$ and using (2.7), we obtain

$$-k_{hs}v^sv_i+k_{ts}v^tv^sv_hv_i+\lambda k_{ts}v^sf_i{}^tu_h$$

$$=(1-\lambda^2)(-k_{hi}+k_{it}v^tv_h-k_{jt}f_h{}^jf_i{}^t),$$

from which, taking skew-symmetric parts in h and i,

$$k_{ts}v^s(f_i{}^t u_h - f_h{}^t u_i) = \lambda(k_{ht}v^t v_i - k_{it}v^t v_h).$$

Transvecting this with v^i, we have (2.8). Substituting (2.8) into (2.12) and (2.13), we have respectively (2.9) and (2.10). This completes the proof of Lemma 2.2.

LEMMA 2.3. *Under the same assumptions as those stated in Lemma 2.2, we have*

$$k_t{}^t = \beta, \tag{2.14}$$

$$\begin{aligned}(1-\lambda^2)\{k_{jt}k_i{}^t + (p+\phi)h_{ji}\} &\\ = (p+\phi)(2p+\phi)(u_j u_i + v_j v_i) + \beta^2 v_j v_i &\\ + \beta(p+\phi)(u_j v_i + u_i v_j). &\end{aligned} \tag{2.15}$$

PROOF. Transvecting (2.10) with f^{ji}, we find

$$k_t{}^t + k_{ts}u^t u^s + k_{ts}v^t v^s = -\beta\lambda^2,$$

from which, using (2.7) and (2.8), we have (2.14).

Differentiating (2.8) covariantly and using (1.2), (1.3) and (2.9), we find

$$\begin{aligned}&(\nabla_i k_{jt})v^t + k_{jt}(-k_{is}f^{ts} + \lambda h_i{}^t)\\ &= \left(\frac{\lambda\phi\beta}{1-\lambda^2}u_i + \frac{v^t\nabla_t\phi}{1-\lambda^2}v_i\right)u_j + (p+\phi)(-h_{it}f_j{}^t - \lambda k_{ij})\\ &+ (\nabla_i\beta)v_j + \beta(-k_{it}f_j{}^t + \lambda k_{ij}),\end{aligned}$$

from which, taking skew-symmetric parts in i and j and using (2.10), we get

$$\begin{aligned}-2k_{jt}k_{is}f^{ts} = (\nabla_i\beta)v_j - (\nabla_j\beta)v_i - 2(p+\phi)h_{it}f_j{}^t &\\ + \frac{1}{1-\lambda^2}(v^t\nabla_t\phi + \lambda\beta^2)(u_j v_i - u_i v_j). &\end{aligned} \tag{2.16}$$

Transvecting (2.16) with v^j and using (2.8) and (2.9), we find

$$(1-\lambda^2)(\nabla_i\beta) = \{(v^t\nabla_t\phi) + \lambda\beta^2 + 2\lambda(p+\phi)(2p+\phi)\}u_i + (v^t\nabla_t\beta)v_i. \tag{2.17}$$

Substituting (2.17) into (2.16) and using (2.10), we obtain

$$\begin{aligned}&(1-\lambda^2)\{k_j{}^s k_{st} + (p+\phi)h_{jt}\}f_i{}^t\\ &= -\lambda\beta\{(p+\phi)u_j u_i + \beta v_j u_i - (p+\phi)v_j v_i\}\\ &-\lambda(p+\phi)(2p+\phi)(u_i v_j - u_j v_i).\end{aligned}$$

Transvecting this $f_h{}^i$, we find

$$(1-\lambda^2)\{k_j{}^s k_{st}+(p+\phi)h_{jt}\}(-\delta_h^t+u_h u^t+v_h v^t)$$
$$=-\lambda\beta\{\lambda(p+\phi)u_j v_h+\lambda\beta v_j v_h+\lambda(p+\phi)v_j u_h\}$$
$$-\lambda^2(p+\phi)(2p+\phi)(u_j u_h+v_j v_h),$$

from which, using (2.3), (2.7) and (2.8), we have (2.15). Thus, Lemma 2.3 is proved.

§3. Complete submanifolds with certain conditions.

In this section, we first prove

THEOREM 3.1. *Let M be a complete submanifold codimension 2 in a $(2n+2)$-dimensional Euclidean space E such that the scalar curvature of M is constant and $fH=Hf$ and there are global unit normals C and D to M which are parallel in the normal bundle, where H is the second fundamental tensor of M with respect to C, f is tensor field of type (1,1) appearing in the induced structure (f, g, u, v, λ) of M. If $\nabla_X\lambda=\phi v(X)$, ϕ being non-zero differentiable function on M, then M is in E, congruent to one of the following submanifolds:*

$$E^{2n},\ S^{2n}(r),\ S^n(r)\times S^n(r),\ S^l(r)\times E^{2n-l}\ (l=1, 2, \cdots, 2n-1),$$
$$S^k(r)\times S^k(r)\times E^{2n-2k}\ (k=1, 2, \cdots, n-1),$$

where, $S^k(r)$ denotes a k-dimensional sphere of radius $r(>0)$ imbedded naturally in E (cf. Theorem 3.2 in [2]).

PROOF. We have from equation (1.5) of Gauss,

$$K_{ji}=(h_t{}^t)h_{ji}-h_{jt}h_i{}^t+(k_t{}^t)k_{ji}-k_{jt}k_i{}^t,$$

from which, using (2.4) and (2.14),

$$K_{ji}=(h_t{}^t-p)h_{ji}+\beta k_{ji}-k_{jt}k_i{}^t,$$

from which, transvecting with g^{ji},

$$g^{ji}K_{ji}=(h_t{}^t-p)h_t{}^t+\beta^2-k_{ji}k^{ji}, \tag{2.18}$$

which gives the scalar curvature of M.

On the other hand, we have from (2.4),

$$h_t{}^t=mp,\quad m=\text{constant}, \tag{2.19}$$

where m being the multiplicity of the eigenvalue p of $h_i{}^h$.

Transvecting (2.15) with g^{ji} and using (2.19), we find

$$k_{ji}k^{ji}=-(p+\phi)mp+\beta^2+2(p+\phi)(2p+\phi). \tag{2.20}$$

Substituting (2.19) and (2.20) into (2.18), we obtain

$$g^{ji}k_{ji}=mp^2(m-1)+mp(p+\phi)-2(p+\phi)(2p+\phi), \tag{2.21}$$

which implies that ϕ is constant because of $g^{ji}K_{ji}=\text{const}$. Therefore, using (2.9) and (2.10), we have

$$k_{jt}f_i{}^t-k_{it}f_j{}^t=0. \tag{2.22}$$

Using Theorem 1.1, we get the results.

Transvecting (1.5) with $u^k v^j u^i v^h$ and using (2.3), (2.7) and (2.8), we have

$$K_{kjih}u^k v^j u^i v^h=\phi(2p+\phi)\ (1-\lambda^2)^2.$$

Hence, the sectional curvature $K(\theta)$ with respect to the section θ spanned by u^h and v^h is given by

$$K(\theta)=-\frac{K_{kjih}u^k v^j u^i v^h}{(u_j u^j)(v_j v^j)}=-\phi(2p+\phi),$$

which shows that if $K(\theta)$ is constant, then ϕ is constant. Thus we see from (2.21) that the scalar curvature of M is constant. Hence, we have

COROLLARY 3.2. *Let M be a complete submanifold of codimension 2 in a $(2n+2)$-dimensional Euclidean space E such that $fH=Hf$ and there are global unit normals C and D to M which are parallel in the normal bundle, where H is the second fundamental tensor of M with respect to C, f is the tensor field of type (1,1) appearing in the induced structure (f, g, u, v, λ) of M. If the sectional curvature $K(\theta)$ with respect to the section spanned by u^h and v^h is constant and $\lambda(1-\lambda^2)$ is non-zero almost everywhere in M, $\nabla_X\lambda=\phi v(X)$, ϕ being non-zero differentiable function, then the same conclusion as in Theorem 3.1 is valid.*

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