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BIQUASI-PROXIMITY SPACES AND COMPACTIFICATION OF A PAIRWISE PROXIMITY SPACE

By M.K. Singal and Sunder Lal

1. Introduction. This paper investigates certain properties of a set X equipped with two quasi-proximities. Such a space is called a biquasi-proximity space and is equivalent to a bitopological space, i.e, every bitopological space admits of two quasi-proximities which yield the topologies of the space. Section 2 contains definitions and results that are needed in order to prove the results in the following sections. In section 3 are given characterizations of some separation axioms in bitopological spaces. These characterizations are purely in terms of the two quasi-proximities of the space. When the two quasi-proximities coincide, they reduce to the corresponding results about quasi-proximity spaces [7]. In section 4 we construct a compactification of a pairwise proximity space. Such spaces are equivalent to pairwise completely regular spaces. This compactification reduces to the Smirnov compactification of a proximity space if the quasi-proximities of pairwise proximity space coincide.

2. Definitions and elementary properties.

(a) Proximities Let X be a non-empty set. A *quasi-proximity* on X is a relation δ on the family $\mathscr{P}(X)$ of all subsets of X satisfying the following axioms:

Q.1. $(A, B)\in\delta$ implies $A\neq\phi$, $B\neq\phi$

Q.2. $(A\cup B, C)\in\delta$ iff $(A, C)\in\delta$ or $(B, C)\in\delta$, and $(A, B\cup C)\in\delta$ iff $(A, B)\in\delta$ or $(A, C)\in\delta$

Q.3. $A\cap B\neq\phi$ implies $(A, B)\in\delta$

Q.4. If $(A, B)\notin\delta$, then there exists an $E\in\mathscr{P}(X)$ such that $(A, E)\notin\delta$ and $(X-E, B)\notin\delta$.

A *proximity* δ on X is a quasi-proximity δ on X, which is also symmetric, i.e.,

Q.5. $(A, B)\in\delta$ implies $(B, A)\in\delta$.

A (quasi-) proximity δ is said to be *separated* if it satisfies the axiom

Q.6. $(\{x\}, \{y\})\in\delta$ implies $x=y$.

To each quasi-proximity δ on X there is associated another quasi-proximity δ^{-1} on X defined by

$$(A, B)\in\delta^{-1} \text{ iff } (B, A)\in\delta.$$

δ^{-1} is called the *conjugate* of δ. Clearly $(\delta^{-1})^{-1}=\delta$ and δ is a proximity iff $\delta=\delta^{-1}$.

A subset C of a quasi-proximity space (X, δ) is said to be *closed* if $(\{x\}, C)\in\delta$ implies $x\in C$. The topology defined by the collection of all such closed sets is called the *topology generated by* δ and is denoted by $\mathscr{T}(\delta)$. This topology need not satisfy any separation axiom, but it is necessarily completely regular if δ is a proximity. Also $\mathscr{T}(\delta)$ is T_1 iff δ is separated. Any topological space $(X, \mathscr{T})$ can be equipped with a quasi-proximity δ such that $\mathscr{T}(\delta)=\mathscr{T}$ [9, 12]. Thus quasi-proximity spaces are equivalent to topological spaces. Likewise biquasi-proximity spaces are equivalent to bitopological spaces.

Let (X, δ) be a proximity space. A collection σ of subsets of X is called a *cluster* if the following conditions are satisfied:

(i) If A and B belong to σ then $(A, B)\in\delta$

(ii) If $(A, B)\in\delta$ for every $B\in\sigma$, then $A\in\sigma$

(iii) If $A\cup B\in\sigma$, then, $A\in\sigma$ or $B\in\sigma$.

For each $x\in X$, the collection

$$\{A\subset X : (A, x)\in\delta\}$$

is a cluster, called a *point cluster*. The concept of cluster is analogue of ultrafilter in topological spaces and is used in constructing Smirnov compactification of a proximity space.

If δ is a quasi-proximity on X, then δ^* defined by "$(A, B)\in\delta^*$ iff for each finite covers $\{A_1, A_2, \cdots A_m\}$, $\{B_1, B_2, \cdots, B_n\}$ of A and B, $(A_i, B_j)\in\delta$ and $(B_j, A_i)\in\delta$ for some $i, =1, \cdots, m$, $j=1, \cdots, n$," is a proximity on X. It is called the *proximity generated by* δ.

Other terms about proximities not defined here can be found in [8].

(b) Bitopological Spaces. If $\mathscr{T}_1$ and $\mathscr{T}_2$ are two topologies on a non-empty set X, then the ordered triple $(X, \mathscr{T}_1, \mathscr{T}_2)$ is called a bitopological space [2]. It is said to be (*a*) *pairwise* T_0 [6] if for every pair of distinct points there exists a $\mathscr{T}_1$ or a $\mathscr{T}_2$-neighbourhood of one point not containing the other; (*b*) *pairwise* T_1 [6] if for every pair of distinct points x, y there exists a $\mathscr{T}_1$ or a $\mathscr{T}_2$-neighbourhood of x not containing y; (*c*) *pairwise* T_2 [14] if for every pair of distinct points x, y there exist a $\mathscr{T}_i$ neighbourhood of x and a $\mathscr{T}_j$-neighbourhood of y $(i \neq j)$ which are disjoint; (*d*) *pairwise Urysohn* [10] if for any two points x and y such that $x \neq y$ there exists a $\mathscr{T}_i$-open set U and a $\mathscr{T}_j$-open set V such that $x \in U$, $y \in V$, $\mathscr{T}_j$-cl$U \cap \mathscr{T}_i$-cl$V = \phi$ $(i \neq j)$: (*e*) *pairwise* R_0 [6] if for every G in $\mathscr{T}_i$, $x \in G \Rightarrow \mathscr{T}_j$-cl$\{x\} \subset G$, $(i \neq j)$: (*f*) *pairwise* R_1 [7] if for $x, y \in X$, and $i \neq j$ $\mathscr{T}_i$-cl$\{x\} \neq \mathscr{T}_j$-cl$\{y\}$ implies x has a $\mathscr{T}_j$-neighbourhood and y has a $\mathscr{T}_i$ neighbourhood which are disjoint; (*g*) *pairwise regular* [2] if every point of X has a $\mathscr{T}_i$-neighbourhood base consisting of $\mathscr{T}_j$-closed sets $(i \neq j)$: (*h*) *pairwise completely regular* [4] if for every $\mathscr{T}_i$- closed set F and $x \in X - F$ and $i \neq j$, there exists a function $f : X \longrightarrow [0, 1]$ which is $\mathscr{T}_i$-upper semi-continuous (u.s.c.) and $\mathscr{T}_j$-lower semi-continuous (l.s.c.) such that $f(x) = 0$, $f(F) = \{1\}$; (*i*) *pairwise normal* [2] if for every $\mathscr{T}_i$-closed set A and a $\mathscr{T}_j$- closed set B with $A \cap B = \phi$, there exists a $\mathscr{T}_j$-neighbourhood F of A and a $\mathscr{T}_i$-neighbourhood G of B such that $F \cap G = \phi$.

A cover $\mathscr{U}$ of a bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be *pairwise open* [1] if $\mathscr{U} \subset \mathscr{T}_1 \cup \mathscr{T}_2$ and if furthermore contains a non-empty member of $\mathscr{T}_i (i=1, 2)$. A bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be *pairwise compact* [3] if each $\mathscr{T}_i$-closed set $C \neq X$ is $\mathscr{T}_j$-compact $(i \neq j)$. A cover $\mathscr{U}$ of a bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be $\mathscr{T}_1 - \mathscr{T}_2$- open [13] if $\mathscr{U} \subset \mathscr{T}_1 \cup \mathscr{T}_2$. If every $\mathscr{T}_1 - \mathscr{T}_2$ open cover of X has a finite subcover, the bitopological space $(X, \mathscr{T}_1, \mathscr{T}_2)$ is said to be *compact* [13]. Clearly every compact bitopological space is pairwise compact.

Notations. In a quasi-proximity space (X, δ), $\mathscr{T}(\delta)$-closure of $A \subset X$ is denoted by δ-cl$A$. $A \Subset B$ means $(A, X-B) \notin \delta$, B is called a δ-neighbourhood of A. $(\{x\}, \{y\}) \in \delta$ is written as $(x, y) \in \delta$. Closure of $\{x\}$ in $\mathscr{T}(\delta_i)$ is written as $\bar{x}^i$.

3. Biquasi-proximity Spaces. In this section we give characterizations of some separation axioms is terms of the quasi-proximities of a biquasi-proximity space, without explicit reference to the induced topologies. Proofs of the results are simple and hence many of them are omitted.

THEOREM 3.1 *A biquasi-proximity space* (X, δ_1, δ_2) *is pairwise* T_0 *iff* $(x, y) \in \beta \cap \beta^{-1} \Rightarrow x = y$, *where* $\beta = \delta_1 \cap \delta_2$.

THEOREM 3.2 (X, δ_1, δ_2) *is pairwise* T_1 *iff* $(x, y) \in \beta \Rightarrow x = y$, *where* $\beta = \delta_1 \cap \delta_2$.

THEOREM 3.3 *A biquasi-proximity space* (X, δ_1, δ_2) *is pairwise* R_0 *iff* $(x, y) \in$

$\delta_i \Leftrightarrow (y, x) \in \delta_j \quad i \neq j$.

THEOREM 3.4 *A biquasi-proximity space* (X, δ_1, δ_2) *is pairwise* T_2 *iff for each pair* x, y *of distinct points of* X *there exists a covering* $\{A, B\}$ *of* X *such that* $(x, A) \notin \delta_i$, $(y, B) \notin \delta_j$ $i \neq j$, $i, j = 1, 2$.

PROOF. The necessity is obviously. For sufficiency, suppose the condition is satisfied, so that for $x \neq y$, there exist A, B such that $(x, A) \notin \delta_i$, $(y, B) \notin \delta_j$, $A \cup B = X$. Clearly $X - A$ and $X - B$ are the required disjoint neighbourhoods of x and y respectively.

THEOREM 3.5 *A biquasi-proximity space* (X, δ_1, δ_2) *is pairwise* T_2 *iff* $((x, y), \Delta) \in \delta \Rightarrow x = y$, *where* $\delta = \delta_i \times \delta_j$ *the product quasi-proximity on* $X \times X$ and $\Delta = \{(x, x); x \in X\}$.

PROOF. (X, δ_1, δ_2) is pairwise T_2 iff Δ is closed in δ iff $((x, y), \Delta) \in \delta \Rightarrow (x, y) \in \Delta \Leftrightarrow x = y$.

THEOREM 3.6 *A biquasi-proximity space* (X, δ_1, δ_2) *is pairwise* R_1 *iff for each pair* x, y *of points of* X *for which there exists a* $p \in X$ *such that*

$$(p, x) \in \delta_i \text{ but } (p, y) \notin \delta_j$$

or

$$(p, x) \notin \delta_i \text{ but } (p, y) \in \delta_j,$$

there exists a cover $\{A, B\}$ *of* X *such that* $(x, A) \notin \delta_i$ *and* $(y, B) \notin \delta_j$.

THEOREM 3.7 *A biquasi-proximity space* (X, δ_1, δ_2) *is pairwise Urysohn iff for* $x \neq y$, *there exists a pairwise open cover* $\{A_i, A_j\}$ *of* X *such that* $(x, A_i) \notin \delta_i$, $(y, A_j) \notin \delta_j$, *where,* A_i *is* δ_i*-open,* A_j *is* δ_j*-open,* $i \neq j$.

COROLLARY 3.1 *For a biquasi-proximity space* (X, δ_1, δ_2), (X, δ_1, δ_2) *is pairwise Urysohn* $\Rightarrow (X, \delta_1, \delta_2)$ *is pairwise* $T_2 \Rightarrow (X, \delta_1, \delta_2)$ *is pairwise* $R_1 \Rightarrow (X, \delta_1, \delta_2)$ *is pairwise* R_0.

THEOREM 3.8 *A biquasi-proximity space* (X, δ_1, δ_2) *is pairwise regular iff for each* $x \in X$ *and for each* δ_i*-neighbourhood* A *of* x, *there exists a* δ_i*-neighbourhood* B *of* x *such that* δ_j cl $B \Subset_i A$, $i \neq j$.

PROOF. The sufficiency part is obvious. For necessity part, let x be in X and let A be a δ_i-neighbourhood of x. Since $(x, X - A) \notin \delta_i$, there exists a δ_i-neighbourhood D of x such that $(D, X - A) \notin \delta_i$. Let B be a δ_j-closed neighbourhood of x contained in D. Then $B \Subset_i A$.

THEOREM 3.9 *A biquasi-proximity space* (X, δ_1, δ_2) *is pairwise completely regular* *iff there exists a quasi-proximity δ on X with $\mathscr{T}(\delta_i)=\mathscr{T}(\delta)$, $\mathscr{T}(\delta_j)=\mathscr{T}(\delta-1)$, $\neq ij$.*

PROOF. See [5].

As is well-known there are pairwise normal spaces which are not pairwise completely regular. Therefore the topologies of a pairwise normal space need not be conjugate. But it they are, we have the following result.

THEOREM 3.10 *(X, δ, δ^{-1}) is pairwise normal iff δ is defined as $(A, B)\in\delta$ iff Q-cl $A\cap P$-cl $B\neq\phi$ where (P, Q) is a pair of conjugate topologies on X.*

PROOF. Clearly $P=\mathscr{T}(\delta)$, $Q=\mathscr{T}(\delta^{-1})$. Now the result follows from [5].

It is known that a compact Hausdorff space admits of a unique proximity. Here we generalize this result to bitopological space.

LEMMA 3.1 *Let (X, P, Q) be a pairwise completely regular space and let δ be a quasi-proximity on X such that $P=\mathscr{T}(\delta)$ $Q=\mathscr{T}(\delta^{-1})$. If A is P-compact and B is P-closed, then $A\cap B=\phi\Rightarrow(A, B)\notin\delta$.*

PROOF. For each $a\in A$, $(a,B)\notin\delta$, so that there exists a P-neighbourhood N_a of a such that $(N_a, B)\notin\delta$. $\{N_a : a\in A\}$ is a P-neighbourhood cover of A, and so admits of a finite subcover $\{N_{a_i} : i=1,\cdots, n\}$. Clearly $(A, B)\notin\delta$, since $A\subset N=\bigcup_{i=1}^{n}N_{a_i}$.

THEOREM 3.11. *For every pairwise compact space (X, P, Q), whose topologies are conjugate there is a unique quasi-proximity δ on X such that $P=\mathscr{T}(\delta)$, $Q=\mathscr{T}(\delta^{-1})$.*

PROOF. Let us define δ on X by setting $(A, B)\in\delta$ iff (Q-cl $A)\cap(P$-cl $B)\neq\phi$. Then δ is a quasi-proximity on X such that $P=\mathscr{T}(\delta)$, $Q=\mathscr{T}(\delta^{-1})$. If δ_1 is any other quasi-proximity with this property, then (Q-cl $A)\cap(P$-cl $B)\neq\phi\Rightarrow$ (Q-cl A, P-cl $B)\in\delta_1\Leftrightarrow(A, B)\in\delta_1$. Since Q-closed sets in a pairwise compact space are P-compact, therefore Q-cl A is P-compact. From lemma 3.1 (Q-cl $A)\cap(P$-cl $B)=\phi\Rightarrow(A, B)\notin\delta_1$. Therefore $\delta_1=\delta$.

4. Compactification of a pairwise proximity space. In this section we shall consider only pairwise T_1 biquasi-proximity spaces.

As we have seen in Theorem 3.9 a biquasi-proximity space (X, δ, δ^{-1}) is pairwise completely regular and all pairwise completely regular spaces are of this form, we shall call such spaces *pairwise proximity spaces.*

DEFINITION 4.1 Let (X, δ_1) and (Y, δ_2) be two quasi-proximity spaces. A function $f: X \longrightarrow Y$ is said to be *proximity mapping* if $(A, B) \in \delta_1 \Rightarrow (f(A), f(B)) \in \delta_2$.

DEFINITION 4.2 Let (X, δ_1, δ_2), $(Y, \delta_1', \delta_2')$ be two biquasi-proximity spaces. A function $f: X \longrightarrow Y$ is said to be *pairwise proximity mapping* if $f: (X, \delta_1) \longrightarrow (Y, \delta_1')$ and $f: (X, \delta_2) \longrightarrow (Y, \delta_2')$ is a proximity mapping.

It is clear that $f: (X, \delta_1) \longrightarrow (Y, \delta_2)$ is a proximity mapping iff $f: (X, \delta_1, \delta_1^{-1}) \longrightarrow (Y, \delta_2, \delta_2^{-1})$ is a pairwise proximity mapping. If δ_i^* is the proximity generated by δ_i, $i=1,2$, then if $f: (X, \delta_1) \longrightarrow (Y, \delta_2)$ is a proximity mapping then so is $f: (X, \delta_1^*) \longrightarrow (Y, \delta_2^*)$.

It is clear that every (pairwise) proximity mapping is (pairwise) continuous. For the converse we have the following result.

THEOREM 4.1 *If $(X, \delta_1, \delta_1^{-1})$ and $(Y, \delta_2, \delta_2^{-1})$ are pairwise proximity spaces and X is pairwise compact, then every pairwise continuous function $f: X \longrightarrow Y$ is a pairwise proximity mapping.*

PROOF. If A and B are subsets of X such that $(A, B) \in \delta_1$ then $(\delta_1^{-1}\text{-cl } A) \cap (\delta_1\text{-cl} B) \neq \phi$, by Theorem 3.10. Therefore $f(\delta_1^{-1}\text{-cl} A) \cap f(\delta_1\text{-cl} B) \neq \phi$. Since f is pairwise continuous, $f(\delta_1^{-1}\text{-cl} A) \subset \delta_2^{-1}\text{-cl}(f(A))$ and $f(\delta_1\text{-cl} B) \subset \delta_2\text{-cl}(f(B))$. Therefore $\delta_2^{-1}\text{-cl}(f(A)) \cap \delta_2\text{-cl}(f(B)) \neq \phi$, which yields $(\delta_2^{-1}\text{-cl}(f(A)), \delta_2\text{-cl}(f(B))) \in \delta_2$, which is equivalent to $(f(A), f(B)) \in \delta_2$ as required.

Let δ be a quasi-proximity on X and let δ^* be the proximity on X generated by δ. Then [11]

$$\mathscr{T}(\delta^*) = \sup\{\mathscr{T}(\delta), \mathscr{T}(\delta^{-1})\}.$$

Therefore (X, δ, δ^{-1}) is compact iff the proximity space (X, δ^*) is compact.

We shall now construct compactification of a pairwise proximity space (X, δ, δ^{-1}), whose construction is similar to that Smirnov compactification given in [8]. Let δ^* be the proximity generated by δ and let $\underset{\sim}{X}$ be the set of all clusters in (X, δ^*). For $A \subset X$, let $\bar{\underset{\sim}{A}} = \{\sigma \in \underset{\sim}{X} : A \in \sigma\}$ and let $f: X \longrightarrow \underset{\sim}{X}$ be defined by setting $f(x) = \sigma_x$, the point cluster. Then (i) f is one-to-one, since δ^* is separated and (ii) $f(A) \subset \bar{\underset{\sim}{A}}$, since $A \in \sigma_x$ for each $x \in A$.

LEMMA 4.1 *There exists a quasi-proximity $\underset{\sim}{\delta}$ on $\underset{\sim}{X}$ such that the pairwise proximity space $(\underset{\sim}{X}, \underset{\sim}{\delta}, \underset{\sim}{\delta}^{-1})$ is compact.*

PROOF. We define $\underset{\sim}{\delta}$ on the power set of $\underset{\sim}{X}$ as follows: For subsets $\underset{\sim}{P}$, $\underset{\sim}{Q}$ of $\underset{\sim}{X}$, we let $(\underset{\sim}{P}, \underset{\sim}{Q}) \in \underset{\sim}{\delta}$ iff $\underset{\sim}{P} \subset \underset{\sim}{\bar{A}}$, $\underset{\sim}{Q} \subset \underset{\sim}{\bar{B}}$ implies $(A, B) \in \delta$. It follows easily that $\underset{\sim}{\delta}$ is a quasi-proximity on $\underset{\sim}{X}$. Let $\underset{\sim}{\delta}^*$ be the proximity on X generated by $\underset{\sim}{\delta}$. Let $\underset{\sim}{\delta}^{**}$ denote the proximity of Smirnov compactification [8]. We shall show that $\underset{\sim}{\delta}^*$ and $\underset{\sim}{\delta}^{**}$ are same. Since $\underset{\sim}{\delta}^*$ is the smallest proximity finer than $\underset{\sim}{\delta}$ and $\underset{\sim}{\delta}^{-1}$ and the proximity $\underset{\sim}{\delta}^{**}$ is finer than both $\underset{\sim}{\delta}$ and $\underset{\sim}{\delta}^{-1}$ therefore δ^{**} is finer than $\underset{\sim}{\delta}^*$. Conversely, assume $(\underset{\sim}{P}, \underset{\sim}{Q}) \notin \underset{\sim}{\delta}^{**}$, then there are subsets A and B of X such that $\underset{\sim}{P} \subset \underset{\sim}{\bar{A}}$, $\underset{\sim}{Q} \subset \underset{\sim}{\bar{B}}$ and $(A, B) \notin \delta^*$ the proximity generated by δ. But $(A, B) \notin \delta^*$ implies the existence of finite covers $\{A_1, \cdots, A_m\}$, $\{B_1, \cdots, B_n\}$ of A and B such that $(A_i, B_j) \notin \delta$, $(B_j, A_i) \notin \delta$ forany $i=1, \cdots, m$, $j=1, \cdots, n$. Clearly $\{\underset{\sim}{\bar{A}}_i : i=1, \cdots, m\}$, $\{\underset{\sim}{\bar{B}}_j : j=1, \cdots, n\}$ are finite covees of $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ and for which $(\underset{\sim}{\bar{A}}_i, \underset{\sim}{\bar{B}}_j) \notin \underset{\sim}{\delta}$ $(\underset{\sim}{\bar{B}}_j, \underset{\sim}{\bar{A}}_i) \notin \underset{\sim}{\delta}$ forany $i=1, \cdots, m$, $j=1, \cdots, n$. This proves that $(\underset{\sim}{P}, \underset{\sim}{Q}) \notin \underset{\sim}{\delta}^*$, showing that $\underset{\sim}{\delta}^{**}$ is coarser than $\underset{\sim}{\delta}^*$. This together with the earlier made observation implies $\underset{\sim}{\delta}^{**} = \underset{\sim}{\delta}^*$. Since thespace $(\underset{\sim}{X}, \underset{\sim}{\delta}^{**})$ is compact [8], $(X, \underset{\sim}{\delta}, \underset{\sim}{\delta}^{-1})$ is compact.

LEMMA 4.2 *(X, δ, δ^{-1}) is pairwise proximally isomorphic to $f(X)$ with the subspace quasi-proximities induced by $\underset{\sim}{\delta}$ and $\underset{\sim}{\delta}^{-1}$ and $f(X)$ is dense in $\underset{\sim}{X}$.*

PROOF. Since closure of $f(X)$ in $\mathscr{T}(\underset{\sim}{\delta}^*)$ is X, $f(X)$ is dense in $\underset{\sim}{X}$. Now $(f(A), f(B)) \in \underset{\sim}{\delta}$ iff $(C, D) \in \delta$, whenever $f(A) \subset \underset{\sim}{\bar{C}}$ $f(B) \subset \underset{\sim}{\bar{D}}$ iff $(C, D) \in \delta$, whenever $A \subset \delta^*$-cl$C$, $B \subset \delta^*$-cl$D$. But this last statement is equivalent to $(A, B) \in \delta$. So that $(f(A), f(B)) \in \underset{\sim}{\delta}$ iff $(A, B) \in \delta$. Thus X is pairwise proximally isomorphic to $f(X)$.

LEMMA 4.3 *Every pairwise proximity mapping g of (X, δ, δ^{-1}) onto a dense subset of a compact space $(Y, \delta_1, \delta_1^{-1})$ extends to a pairwise proximity isomorphism $\bar{g}$ of $(\underset{\sim}{X}, \underset{\sim}{\delta}, \underset{\sim}{\delta}^{-1})$* onto $(Y, \delta_1, \delta_1^{-1})$.

PROOF. If σ is a cluster in X, there corresponds a cluster σ_r' in Y. Since Y is compact, σ' is a point cluster. Thus every point in Y determines a unique cluster (via proximity isomorphism of the dense subspace) in X. Thus there exists a one-to-one map $\bar{g} : \underset{\sim}{X} \longrightarrow Y$, which extends g.

To prove that $\bar{g}$ is a pairwise proximity isomorphism it is sufficient to show that $\bar{g} : (\underset{\sim}{X}, \underset{\sim}{\delta}) \rightarrow (Y, \delta_1)$ is a proximity isomorphism. Let $\underset{\sim}{P}$, $\underset{\sim}{Q}$ be subsets of X such that if$(\underset{\sim}{P}, \underset{\sim}{Q}) \in \underset{\sim}{\delta}$, then $(\underset{\sim}{\delta}^{-1}$-cl $\underset{\sim}{P}) \cap (\underset{\sim}{\delta}$-cl $\underset{\sim}{Q}) \neq \phi$. Hence there exists a $\sigma \in \underset{\sim}{X}$ such that $(\underset{\sim}{P}, \{\sigma\}) \in \underset{\sim}{\delta}$ and $(\{\sigma\}, \underset{\sim}{Q}) \in \underset{\sim}{\delta}$. Let $y = \bar{g}(\sigma)$, then we have $(\{y\}, \bar{g}(\underset{\sim}{Q})) \in \delta_1$ and $(\bar{g}(\underset{\sim}{P}), \{y\}) \in \delta_1$ whence $(\bar{g}(\underset{\sim}{P}) \bar{g}(\underset{\sim}{Q})), \in \delta_1$. Conversely consider $(\bar{g}(\underset{\sim}{P}),$

$\bar{g}(\underset{\sim}{Q}))\in\delta_1$, then $(\delta_1^{-1}\text{-cl}\bar{g}(\underset{\sim}{P}))\cap(\delta_1\text{-cl}\bar{g}(\underset{\sim}{Q}))\neq\phi$ since Y is compact and hence pairwise compact. Let y be in this intersection and let $\sigma=\bar{g}^{-1}(y)$. If $A\in\sigma$ and $\underset{\sim}{P}\subset\underset{\sim}{\bar{B}}$, then $(\bar{g}(\underset{\sim}{P}),A)\in\delta$ and $\bar{g}(\underset{\sim}{P})\subset\delta^*\text{-cl}B$, which imply $(B,A)\in\delta$ so that $(\underset{\sim}{P},\{\sigma\})\in\underset{\sim}{\delta}$. Similarly $(\{\sigma\},\underset{\sim}{Q})\in\underset{\sim}{\delta}$ from which we conclude that $(\underset{\sim}{P},\underset{\sim}{Q})\in\underset{\sim}{\delta}$.

Above three lemmas taken together prove the main results of this section.

THEOREM 4.2 *Every pairwise proximity space (X,δ,δ^{-1}) is a dense subspace of a unique (up to pairwise proximity isomorphism) compact space $(\underset{\sim}{X},\underset{\sim}{\delta},\underset{\sim}{\delta}^{-1})$. If A, B are subsets of X, then $(A,B)\in\delta$ iff $(\underset{\sim}{\delta}^{-1}\text{-cl}A)\cap(\underset{\sim}{\delta}\text{-cl}B)\neq\phi$.*

REMARK In the statement of the above theorem, we have identified X with $f(X)$ as is usually done.

We conclude this section by an important result about the extension of maps from the spaces to their compactifications.

THEOREM 4.3 *Every pairwise proximity mapping g of $(X,\delta_1,\delta_1^{-1})$ onto $(Y,\delta_2,\delta_2^{-1})$ has a unique extension $\bar{g}$ which is pairwise proximity mapping of the compactification of X onto the compactification of Y.*

PROOF. If σ_1 is a cluster in X, then there corresponds a cluster σ_2 in Y (since $g:(X,\delta_1^*)\to(Y,\delta_2^*)$ is a proximity mapping) such that

$$\sigma_2=\{P\subset Y:(P,g(C))\in\delta_2^*\text{ for all }C\in\sigma_1\}.$$

Let $\bar{g}(\sigma_1)=\sigma_2$. Then $\bar{g}$ maps $\underset{\sim}{X}$ to $\underset{\sim}{Y}$. Clearly $\bar{g}(\sigma_x)=\sigma g(x)$, i.e. $\bar{g}$ agrees with g on X (identifying X with $f(X)$). To show that $\bar{g}$ is a pairwise proximity mapping, we shall show that $(\underset{\sim}{P},\underset{\sim}{Q})\in\underset{\sim}{\delta}_1$ implies $(\bar{g}(\underset{\sim}{P})),\bar{g}(\underset{\sim}{Q}))\in\underset{\sim}{\delta}_2$, i.e., if $\bar{g}(P)\subset\underset{\sim}{\bar{A}}$, $\bar{g}(\underset{\sim}{Q})\subset\underset{\sim}{\bar{B}}$, then $(A,B)\in\delta_2$. If $(A,B)\notin\delta_2$, then there exists sets C and D in Y such that $(A,Y-C)\notin\delta_2$, $(Y-D,B)\notin\delta_2$ and $(C,D)\notin\delta_2$. Since $\bar{g}(\underset{\sim}{P})\subset\underset{\sim}{\bar{A}}$, $Y-C$ belongs to no cluster in $\bar{g}(\underset{\sim}{P})$. For if $Y-C$ is in some cluster in $\bar{g}(\underset{\sim}{P})$, then A and $Y-C$ belong to some cluster and so $(A,Y-C)\in\delta^*$, which is a contradiction.

So $\overline{g^{-1}(Y-C)=X-g^{-1}(C)}$ belongs to no cluster in $\underset{\sim}{P}$. This shows that $\underset{\sim}{P}\subset\underset{\sim}{g}^{-1}(c)$. Similarly $Q\subset\underset{\sim}{g}^{-1}(D)$. Since $(\underset{\sim}{P},\underset{\sim}{Q})\in\delta_1$, we must have $(g^{-1}(C),g^{-1}(D))\in\delta_1$ which yields a contradiction, since g is a proximity mapping from (X,δ_1) onto (Y,δ_2). Theorefore $\bar{g}:(\underset{\sim}{X},\underset{\sim}{\delta}_1)\to(\underset{\sim}{Y},\underset{\sim}{\delta}_2)$ must be a proximity mapping.

That $\bar{g}$ is onto follows from the fact that $f(Y)\subset\bar{g}(\underset{\sim}{X})\subset\underset{\sim}{Y}$, $f(Y)$ is dense in $\underset{\sim}{Y}$ and $\bar{g}(\underset{\sim}{X})$ is compact w.r.t. $\underset{\sim}{\delta}_1^*$.

We now show that $\bar{g}$ is unique. Suppose $\bar{g}'\neq\bar{g}$ is another extension. Then there is a $\sigma\in\underset{\sim}{X}$ such that $\bar{g}'(\sigma)\neq\bar{g}(\sigma)$. Since $\underset{\sim}{Y}$ is Hausdorff w.r.t $\mathscr{T}(\underset{\sim}{\delta}_2^*)$ and

$\bar{g} : (\underset{\sim}{X}, \mathscr{T}(\underset{\sim}{\delta}_1{}^*) \to (\underset{\sim}{Y}, \mathscr{T}(\underset{\sim}{\bar{\delta}}_2{}^*)))$ is continuous there exists a neighbourhood $\underset{\sim}{E}$ of σ such that $\bar{g}(\underset{\sim}{E}) \cap \bar{g}'(\underset{\sim}{E}) = \phi$, Since $f(X)$ is dense in $\underset{\sim}{X}$, there exists $x \in X$ such that $\sigma_x \in \underset{\sim}{E} \cap f(X)$. For such a σ_x, $\bar{g}(\sigma_x) \neq \bar{g}'(\sigma_x)$. Therefore g and $\bar{g}'$ do not agree on $f(X)$ and hence not on X.

Meerut University
Meerut-1 (U.P.) India

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