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LIFTING TENSOR FIELDS AND CONNECTIONS TO TANGENT BUNDLES[1]

SHIGERU ISHIHARA

The differential geometry of tangent bundles of Riemannian manifolds studied by Sasaki [7]. Yano and Kobayashi [9] studied prolongation of tensor fields and connections to tangent bundles. In the present note, we give a brief sketch of the theory of prolongation of tensor fields and connections to tangent bundles.

Let M be an n-dimensional differenctiable manifold. Denote by $T(M)$ the tangent bundle over M and by π: $(TM)\to M$ the bundle projection of $T(M)$ (Manifolds, functions, vector fields, tensor fields and connections we discuss are assumed to be differentiable and of class C^∞.) For a function f in M, we put $f^V=f\cdot\pi$ and call it the vertical lift of f. Since the exterior differential df of f is a 1-form in M, df can be naturally considered as a function in $T(M)$, which is denoted by f^C and called the complete lift of f. For a vector field X in M, we define two vector fields X^V and X^C in $T(M)$ in such a way that $X^V f^C=(Xf)^V$, f being an arbitrary function in M, and exp $X^C=d(\exp X)$. We call X^V and X^C the vertical and the complete lifts of X, respectively. For a 1-form ω in M, we define two 1-forms ω^V and ω^C in $T(M)$ in such a way that $\omega^V(X^C)=\omega(X)^V$ and $\omega^C(X^C)=\omega(X)^C$, X being an arbitrary vector field in M. We call X^V and X^C the vertical and the complete lifts of X, respectively. The following formulas are easily verified:

$$
(1)\qquad \begin{aligned}
&(fg)^V=f^Vg^V, && (fg)^C=f^Cg^V+f^Vg^C,\\
&(fX)^V=f^VX^V && (fX)^C=f^CX^V+f^VX^C,\\
&(f\omega)^V=f^V\omega^V, && (f\omega)^C=f^C\omega^V+f^V\omega^C,
\end{aligned}
$$

where f, g are arbitrary functions, X an arbitrary vector field and ω an arbitrary 1-form, all in M. Taking account of formulas (1), we can define inductively the vertical lift T^V and the complete lift T^C of a tensor field T of arbitrary type in M by using the following formulas:

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$$(2)\qquad \begin{aligned} &(S\otimes T)^V = S^V\otimes T^V, && (S\otimes T)^C = S^C\otimes T^V + S^V\otimes T^C \\ &(S+T)^V = S^V + T^V, && (S+T)^C = S^C + T^C \end{aligned}$$

for any tensor fields S and T in M.

Given a pseudo-Riemannian metric g in M. Then its complete lift g^C is a pseudo-Riemannian metric in $T(M)$ with n positive and n negative signs, where dim $M=n$. If F is an almost complex structure in M, then its complete lift F^C is also an almost complex structure in $T(M)$. The almost complex structure F^C in $T(M)$ is integrable if and only if the almost complex structure is so in M.

Let ∇ be an affine connection in M. Then there is a unique affine connection ∇^* in $T(M)$ satisfying $\nabla^*_{X^C} Y^C = (\nabla_X Y)^C$ for any vector field X and Y in M. We call ∇^* the lift of ∇ to $T(M)$. The curvature tensor R^* and the torsion tensor T^* of ∇^* coincide with the complete lifts R^C and T^C respectively, where R and T denote respectively the curvature and the torsion tensors of ∇. For any tensor field T, we have the formula

$$(3)\qquad \nabla^* T^C = (\nabla T)^C$$

Thus, ∇^* is the Levi-Civita's connection in the pseudo-Riemannian manifold $(T(M),\ g^C)$, if ∇ is the Levi-Civita's connection of a pseudo-Riemannian metric g in M. Taking account of (3), we have $\nabla^* R^* = (\nabla R)^C$. Therfore the pseudo-Riemannian manifold $(T(M), g^C)$ is locally symmetric if the pseudo-Riemannian manifold (M, g) is so. (We can prove that $(T(M),\ g^C)$ is symmetric if (M, g) is so.)

Let G be a Lie group with group multiplication μ: $G\times G\to G$. Then the differential mapping $d\mu$: $T(G\times G)\to T(G)$ of μ defines a group multiplication in $T(M)$ if $T(G\times G)$ is naturally identified with $T(G)\times T(G)$. Then $T(G)$ becoms a Lie group with group multiplication $d\mu$: $T(G)\times (T(G)\to T(G)$ and is called the tangent group to G. If $X_1, \cdots, X_r$ are vector fields in G which form a basis of the Lie algebra of G, then the lifts $X_1{}^V, \cdots, X_r{}^V, X_1{}^C, \cdots, X_r{}^C$ form a basis of the Lie algebra of the tangent group $T(G)$.

If Φ is the holonomy group of an affine connection ∇ in M, then the tangent group $T(\Phi)$ to Φ is the holonomy group of the lift ∇^* of ∇ to $T(M)$.

Generalizing the arguments above, we can develope the theory of prolongation of G-structures to tangent bundles or those of higher order (Morimoto [**1, 2, 3, 4, 5, 6**] and Yano and Ishihara [**9**]).

References

[1] A. Morimoto, *Prolongations of G-structures to tangent bundles*, Nagoya Math. J., **32**(1968), 67-108.

[2] ———, *Prolongations of G-structures to tangent bundles of higher order*, Nagoya Math. J., **38**(1970), 153-179.

[3] ———, *Liftings of some type of tensor fields and connections to tangent bundles of p^V-velocities*, Nagoya Math. J., **40**(1970), 13-31.

[4] ———, *Prolongations of connections to tangent fibre bundles of higher order*, Nagoya Math. J., **40**(1970), 85-97.

[5] ———, *Lifting of tensor fields and connections to tangent bundles of higher order*, Nagoya Math. J., **40**(1970), 99-120.

[6] ———, *Prolongations of G-structure to tangent bundles of higher order*, Nagoya Math. J., **38**(1970), 153-179.

[7] S. Sasaki, *On the differential geometry of tangent bundles of Riemannian manifolds*, Tôhoku Math. J., **10**(1958), 338-364.

[8] K. Yano and S. Kobayashi, *Prolongation of tensor fields and connections to tangent bundles*, I, II, III, J. Math. Soc. Japan, **18**(1966), 144-210, 236-246, **19** (1967), 486-488.

[9] K. Yano and S. Ishihara, *Prolongation of tensor fields and connections to tangent bundles of order* 2, Kōdai Math. Sem. Rep., **20**(1968), 318-354.

[10] ———, *Differential geometry of tangent and cotangent bundle*, Marcel Dekker, New York, to appear.

Tokyo Institute of Technology