Bull. Korean Math. Soc.
Vol. 10 (1973), pp. 21-29

STRUCTURE OF DOOR SPACES

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1. Introduction

This note is a presentation of structure theory for *door spaces*, i.e., spaces in which every subset is either open or closed. We visualize open subsets of a door space by proving that a space X is a minimal door space if and only if either (1) nonempty closed subsets of X form a fixed ultrafilter on X, or (2) nonempty open subsets of X form an ultrafilter (fixed or free) on X. It is an immediate consequence of this result that a door space is minimal exactly when it is connected. For the maximal case, this latter characterization has a counterpart: a Hausdorff door space is maximal if and only if it is extremally disconnected. Finally, assuming the continum hypothesis, we construct a nonmaximal Hausdorff door space of cardinality of the continuum that can be imbedded in the Stone-Cech compactification, βD, of any infinite discrete space D. This is interesting because maximality of X agrees with imbeddability in βD for infinite discrete D if X is a nontrivial Hausdorff door space of denumerable cardinality.

For terminology and notation, we follow [**2**]. Thus, by a *door topology* for a set X, we mean a topology with which X becomes a door space. A space X is *minimal* or *maximal* door space if its topology is minimal or maximal among nontrivial (=not discrete) door topologies for X. As in [**2**], the word "point" will have double senses so as to mean a set with one point as well, and p will stand for $\{p\}$ if p is a point of X.

2. Structure of minimal door spaces

We begin by listing some lemmas, the first of which is of trivial nature.

LEMMA 1. *Every subspace of a door space is a door space.*

LEMMA 2. *If a door space is expressed as the union of two disjoint open* *sets, at least one of them is discrete.*

Proof. Let X be a door space which is the disjoint union of open subsets A and B. If A is not discrete, then it has a point p that fails to be open. For any subset C of B, the set $p \cup C$ can not be open for otherwise p would be open in X. Accordingy, $p \cup C$ is closed in X and C must be closed in B. That is, every subset of B is closed in B implying that B must be discrete.

LEMMA 3. *If a point p of a door space X is not open, then eveery neighborhood as well as every deleted neighborhood of p is open.*

Proof. Let U be an arbitrary neighborhood of p. Since p is interior to U, it is enough for our purpose to prove that $U-p$ is open. To this end, suppose that $U-p$ not open. Since $X-(U-p)=(X-U) \cup p$ must be open in this case, we are led to the contradiction that $p=U \cap ((X-U) \cup p)$ should be open. This completes the proof of Lemma 3.

LEMMA 4. *Let X be a door space and let p be a point of X that fails to be open. If deleted neighborhoods of p do not form an ultrafilter on $X-p$, then $X-p$ is an an open discrete subspace of X.*

Proof. Express $X-p$ as the union of disjoint sets A and B non of which is a deleted neighborhood of p. This is possible because the filter of deleted neighborhoods of p is not maximal. In this case, neither A nor B can be closed in X for otherwise at least one of A and B would be a delected neighborhood of p. That is, both A and B are open in X. By Lemma 1, $A \cup B$ is a door space and, by Lemma 2, at least one of A and B, say A, is discrete. In order to prove that B is also discrete, let C be any subset of B. Since p must be in the closure of A, $A \cup C$ can not be closed, i.e., $A \cup C$ is open in X. Thus $C=B \cap (A \cup C)$ is open, and B is discrete. We have proved Lemma 4.

We are now ready to state and prove the proposed structure theorem for minimal door spaces:

THEOREM 1. *A space X is a minimal door space if and only if it satisfies one of the following conditions:*

(1) *nonempty closed subsets of X form a fixed ultrafilter on X, or*

(2) *nonempty open subsets of X form an ultrafilter on X.*

Proof. For the if part, let A be any subset of X. Since either of the conditions (1) and (2) implies that exactly one of A and $X-A$ is open, X is a door space. This also proves that X is a minimal door space because the definition of a door space merely requires that at least one of A and $X-A$ is open. To be precise, let X' denote the set X epuipped with a topology properly weaker than the one originally given for X, and let A be an open subset of X that is not open in X'. If A is closed in X', then $X-A$ would be open in X, a patent impossibility. Thus, X' is not a door space, that is, X is a minimal door space.

Conversely, suppose that X is a minimal door space and let p be a point of X that is not open in X. Our argument will be divided into two complementary cases, the first of which is the case when the deleted neighborhoods filter of p is not maximal in $X-p$.

Case 1. If deleted neighborhoods of p fail to form an ultrafilter on $X-p$, every subset of $X-p$ is open by Lemma 4. They are the only proper open subsets of X since X together with all subsets of $X-p$ already form a door topology for the set X. Thus, nonempty closed subsets of X are exactly those subsets of X containing the point p, and they do form a fixed ultrafilter on X.

Case 2. If deleted neighborhoods of p form an ultrafilter on $X-p$, then all these deleted neigborhoods as well as all neighborhoods of p form an ultrafilter on X. By Lemma 3, all members of the latter filter are open. Because these sets together with the empty set make X into a door space, minimality of X implies that they are the only nonempty open subsets of X.

REMARK. Minimal door space can occur only if X has at least two points. Note also that a minimal door space is a T_1-space if and only if nonempty open subsets form a free ultrafilter, in which case X must have infinitely many points.

Suppose that a space X is the union of disjoint nonempty subsets A and B. If X is connected then at most one of A, B can be open, while if X is a door space then at least one of A, B must be open. Accordingly, a connected door space is necessarily a minimal door space as shown by Y. Kim [2]. Theorem 1 provides us with a straightforward proof of the converse of this result. Namely, if X is a minimal door space then either nonempty open subsets form a filter on X or nonempty closed subsets form a filter on X. Hence, X is not the union of two nonempty disjoint subsets both open or both closed in X. We have proved

THEOREM 2. *A door space is a minimal door space if and only if it is connected.*

3. Structure of general door spaces

Let X be a door space and let A be a subset of X that is open but not closed in X. If a topology for X weaker than the one originally given to X forces A to be nonopen, then neither A nor $X-A$ can be open relative to the weaker topology. Since this implies that A must be open relative to any door topology comparable with the original topology of X, we see that the intersection of any chain of door topologies for the set X is again a door topology. Thus, by Zorn's lemma, we have the following result.

LEMMA 5. *Every door topology for a nondegenerate set X contains a minimal door topology for X.*

Roughly speaking, this lemma tells us that any door space can be obtained from a minimal door space by assigning more open sets to it. At this stage, it will be convenient to describe the types of minimal door spaces in more familiar fashion. The first is the case when condition (1) of Theorem 1 is valid, while the second case of Theorem 1 reduces into two subcases according to as the filter of nonempty open sets is fixed or free. Namely, there are following types of minimal door space X:

(1) there is a point p of X such that a proper subset of X is open if and only if it does not contain p,

(2) there is a point q of X such that a nonempty subset of X is open if and only if it contains q,

(3) nonempty open subsets of X form a free ultrafilter Φ.

We shall denote by (X, p), $(X, (q))$ and (X, Φ) the door spaces of types (1), (2) and (3) described above. It is clear that, for any pair of points p and p' in X, any bijection of X to itself carrying p to p' is a homeomorphism between (X, p) and (X, p'). That is, upto homeomorphism, minimal door space structure of type (X, p) is unique on X. Similarly, topological type of $(X, (q))$ is independent from the choice of q. Except for these two types, all minimal door spaces are T_1-spaces: there are enough types of such spaces as shown in [2].

If X and X' are spaces with the same underlying set, we say that X refines X' whenever the topology of X is finer than that of X'. In order to classify the types of door spaces, we must look for open sets in spaces refining a minimal door space. For spaces X refining (X, p), we do this in the following way. Except for the trivial case where X is discrete, X has a unique point that is closed but not open, namely, the point p. Since all other points are open, topology of X is completely determined by the filter Ψ consisting of all deleted neighborhoods of p. If Ψ is a free filter, then X is Hausdorff. If Ψ is not free, the intersection Y of all neighborhoods of p has at least two points. It is immediate that Y is a minimal door space (Y, p), and X can not be a T_1-space. One also easily verifies that $X-Y$ is discrete and open, while $(X-Y) \cup p$ is a nontrivial Hausdorff door space with base point p. Here, of course, "nontrivial" means "not discrete". Thus, if X is not Hausdorff but refines (X, p) properly, then either it is the free union of the minimal door space (Y, p) with the discrete space $X-Y$ or it can be obtained by identifying the base points of the minimal door space (Y, p) and the Hausdorff door space $(X-Y) \cup p$. Note, finally, that a space X refining (X, p) is maximal if and only if the deleted neighborhoods filter Ψ is maximal. In case Ψ is a free ultrafilter on $X-p$, X is essentially a subspace of $\beta(X-p)$ by the Gelfand-Kolmogoroff theorem [1]. If Ψ is fixed, there is a point $q(\neq p)$ such that a nonempty subset of X is open if and only if it contains q or is contained in $X-p$. If, in addition, X has more than two points, then of course X is the free union of a discrete space with the minimal door space $Y=\{p, q\}$ of type $(Y, p)=(Y, (q))$. It seems that this type of spaces escaped from the attention of Yewky Kim [2]. Despite his errornous remarks about the maximal case, however, all his formally stated results are correct except that X should be assumed to be Hausdorff in the corollary to his Theorem 6.

Now let X be a space refining X' which is of type $(X, (q))$ or (X, Φ). If an open subset A of X is not open in X', every superset of $X-A$ is open in X' because nonempty open subsets of X' form a filter. Thus every subset of A is open in X and, in particular, each point of A is open. Therefore, if we let Y denote the union of all open subsets of X that are not open in X', then Y is a discrete subspace of X. If X' is of type $(X, (q))$, then X is either a discrete space or the free union of the dicrete space Y with the minimal door space $(X-Y, (q))$ according to whether $Y=X-q$ or not. Of course, X can not be a T_1-space unless it is discrete. Note also that X is a maximal door space if and only if $X-Y$ has exactly one point p other than q, in which case X refines (X, p) as well. For the only remaining case where X refines (X, Φ), we observe that $X-Y$ is either open or a discrete subset of X without interior. If $X-Y$ is open, either $X=Y$ is discrete or $X-Y$ is minimal of type $(X-Y, \Phi')$ where Φ' is the free ultrafilter on $X-Y$ consisting of those members of Φ failing to meet Y. If $X-Y$ is not open, members of Φ contained in Y form a free ultrafilter Φ' on Y. Accordingly, for each point p of $X-Y$, $Y \cup p$ is a maximal Hausdorff door space whose topological type does not depend on the choice of p. Note that X is maximal exactly when $X-Y$ has only one point p, in which case X is a Hausdorff space refinining (X, p). Otherwise, X is T_1 but not Hausdorff.

Summerizing, since every topology finer than a door topology is again a door topology, we have

THEOREM 3. *Every Hausdorff door space X with base point p can be obtained from the minimal door space (X, p) by assigning a free filter on $X-p$ to serve as the system of deleted open neighborhoods of p. Moreover, X is a maximal door space if and only if this filter is an ultrafilter.*

THEOREM 4. *If a door space X is not Hausdorff, then either*

(1) *X is a minimal door space, or*

(2) *X is the free union of a discrete space with a minimal door space, or*

(3) *X contains a Hausdorff door space Y with base point p such that $X-(Y-p)$ is minimal of type $(X-(Y-p), p)$, or*

(4) *X contains a maximal Hausdorff door space Y with base point p such that $X-Y$ is a non-open discrete set with at least one point and, for each point q of $X-Y$, there is a homeomorphism of Y onto $(Y-p) \cup q$ leaving $Y-p$ pointwise fixed.*

The space X is a maximal door space if it has just two points or the mini-

mal door space part has just two points in the second of above cases.

Once these results are obtained, one can make many remarks concerning door spaces. Here, we list a few of them.

COROLLARY 1. *A door space is a minimal door space if and only if it does not have a point that is both open and closed.*

This, of course, we could have established earlier by using Lemma 2 and Theorem 2.

COROLLARY 2. *If X is a finite set with n points, $n>1$, there are $2n-2$ inequivalent door topologies on X.*

To see that this is the case, observe that if one picks k points of X, $k>1$, he has determined one or two types of nontrivial door topologies on X according to whehter $k=2$ or not.

4. Maximal door spaces

In this section, we look for properties of spaces discriminating maximal ones among door spaces. One of such criteria for door space X with more than two points is that X is maximal or not according to whether X refines exactly two or one minimal door space. If X is a non-Hausdorff door space with more than one point, this is the same as saying that all but just two of the points of X are simultaneously open and closed. For Hausdorff case, we have the following counterpart of Theorem 2.

THEOREM 5. *A Hausdorff door space X with base point p is a maximal door space if and only if it is extremally disconnected.*

The reader may recall here that a space X is extremally disconnected if every open subset of X has closure open in X.

Proof. Suppose that X is maximal and let U be any open subset of X. If U is not closed, it must have $U\cup p$ as closure. Since $X-U$ can not be open but contains p, it follows that $(X-p)-U$ is not a deleted neighborhood of p by Lemma 3. Because deleted neighborhoods of p must form an ultrafilter on $X-p$, we see that $U\cup p$ is a neighborhood of p, which is open again by Lemma 3. Conversely, if X is not maximal, there is a subset U of $X-p$ such that none of U, $(X-p)-U$ is a deleted neighborhood of p. Since $X-$

U can not be open in this case, U has $U \cup p$ as closure. However, $U \cup p$ is not open because p is not an interior point of it. Since U is open as it misses p, this completes the proof.

For spaces with countably many points, we also have

THEOREM 6. *If X is a nontrivial Hausdorff door space of denumerable cardinality whose base point is designated by p, then X is a maximal door space exactly when it can be imbedded in βD for some infinite discrete space D.*

Proof. The "only if" part is almost trivial because X imbeds in $\beta(X-p)$ whenever X is maximal Hausdorff; this is true even without any restriction imposed on the cardinality of X (see [2] for detail). If, conversely, X is imbedded in βD as subspace, then $X-p$ is clossed in $D \cup (X-p)$. But $D \cup (X-p)$ is normal because denumerability of X implies that every open cover of $D \cup (X-p)$ can be refined by σ-locally finite ones. Accordingly, $X-p$ has closure $\beta(X-p)$ in $\beta D = \beta(D \cup (X-p))$. It follows from this that deleted neighborhoods of p in X form a free ultrafilter on $X-p$ as the point p must be in $\beta(X-p)-(X-p)$. This completes the proof of Theorem 6.

Unfortunately, this result is no longer valid for higher cardinalities if the continuum hypothesis is true. We demonstrate this in the following counter example.

EXAMPLE. Let N be the discrete space of natural numbers and let p be a P-point of $\beta N-N$. By this, we of course mean that every G_δ-subset of $\beta N-N$ is a neighborhood in $\beta N-N$ of p. As in [1], $\beta N-N$ has a dense subset of P-points if the continuum hypothesis is true. Again if the continuum hypothesis is true, we may choose open sets U_α indexed by countable ordinals α to form a local basis at p. Since p is a P-point, we may also suppose that this basis is nested. Pick a P-point x_α from each U_α and let X denote the subspace of $\beta N-N$ consisting of the points x_α and p. Since $\alpha < \gamma$ implies $x_\gamma \in U_\alpha$, each neighborhood of p must contain all but a countable number of points x_α. Therefore, deleted neighborhoods in X of p fails to form an ultrafilter since there are uncountably many points x_α. To prove that X is a door space, observe that each point x_α has a compact neighborhood V in $\beta N-p$. Since V contains at most countably many point of X while x_α is a P-point of $\beta N-N$, it follows that x_α is open in X. We have shown that the subspace

X of βN is a Hausdorff door space with base point p. Nevertheless, it is not a maximal door space as we have seen that the deleted neighborhoods filter of p is not maximal.

References

[1] L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, 1960.
[2] Yewky Kim, *Door topologies on an infinite set*, J. Korean Math. Soc., 7 (1970), 49-53.

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