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A NOTE ON SUBALGEBRAS OF $C^*(\mathrm{N})$

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1. Introduction

Let X be a compact Hausdorff space. The present article starts by observing that there is a natural one to one correspondence between upper semicontinuous decompositions of the space X and closed subrings of the function ring $C^*(X)$ which contains all constant functions. Using this fact, we construct explicit examples of function rings on the n-sphere to show that monotone union or intersection of isomorphic rings need not be isomorphic with given ones.

Interpretation of these examples as subalgebras of products of real fields may be of some interest. Infinite (complete) products of reals are discriminated from finite products by the fact that they do not enjoy the monotone union property or weak monotone intersection property on isomorphic subalgebras.

2. Definitions and notation

Before embarking the results, we must recall some definitions that will be of frequent use throughout this paper. If X is a completely regular Hasudorff space, $C^*(X)$ will denote as usual the ring of bounded continuous real functions on X metrized by means of the uniform norm $\|f\|=\sup|f(x)|$. An *analytic subring* is a closed subring A of $C^*(X)$ such that all constant functions belong to A and $f^2 \in A$ implies $f \in A$.

Let X and Y be spaces and let C be any collection of continuous maps from X into Y. A *stationary set* of C is a subset of X on which every function in C is constant. A stationary set which is not a proper subset of any stationary set is called a *maximal stationary set*. Evidently, every maximal stationary set is closed, and X is the sum of maximal stationary sets.

If G is a decomposition of a space X, X/G will denote the quotient space of G. A decomposition G of a Hausdorff space X into compact sets is said to be *upper semicontinuous* provided that, for each open subset U of X, the sum of members of G contained in U is open in X. G is said to be monotone if each member of G is connected.

We denote the Euclidean n-space by E^n. The n-sphere is the subspace $\{x\in E^{n+1}: \|x\|=1\}$ of E^{n+1}, and is denoted by S^n.

3. Closed subrings of $C^*(X)$

In this section, we state a result which, in fact, is equivalent with [**1**, 16E–2]. We decided, however, to include our own proof of it as we believed that the point of view is somewhat different in the present case.

LEMMA 1. *If a subring of $C^*(X)$ is isomorphic with $C^*(Y)$ for some space Y, then it is a closed subring of $C^*(X)$.*

This lemma follows from the fact that every isomorphism of $C^*(Y)$ into $C^*(X)$ is norm preserving, and is an isometry from the complete metric space $C^*(Y)$ into the metric space $C^*(X)$.

LEMMA 2. *If C is a collection of continuous maps of a compact Hausdorff space X into a Hausdorff space Y, then the collection G of maximal stationary sets of C is an upper semicontinous decomposition of X.*

Proof. Obviously, union of any two overlapping stationary sets of C is again a stationary set of C, and G is a decomposition of X. Hence, we need only prove that the quotient space X/G is a Hausdorff space by virtue of [**2**, Theorem 3–31]. To this end, let $p(x)$ and $p(y)$ be distinct points of X/G. where $x, y\in X$ and p is the projection map of G. Since x and y must lie in distinct members of G, there is a function f in C with $f(x)\neq f(y)$. Let U and V be disjoint open neighborhoods of $f(x)$ and $f(y)$, respectively. Since f must be stationary on all members of G, each of the sets $f^{-1}(U)$, $f^{-1}(V)$ is an inverse image under the projection p of a subset of X/G. This then implies that $pf^{-1}(U)$ and $pf^{-1}(V)$ are disjoint open sets containing $p(x)$ and $p(y)$, respectively, because $f^{-1}(U)$ and $f^{-1}(V)$ are open in X by continuity of f. We have completed the proof of Lemma 2.

The following is an alternate version of [**1**, 16E–2].

PROPOSITION 1. *Let X be a compact Hausdorff space. For each closed subring A of $C^*(X)$, the maximal stationary sets of A form an upper semi-continuous decomposition of G of X with $C^*(X/G)$ isomorphic to A and,* *conversely, for each upper semicontinuous decomposition G of X there is one and only one closed subring A of $C^*(X)$ such that A contains all constant functions in $C^*(X)$ and G is the collection of stationary sets of A. Moreover, the closed subring A is analytic if and only if corresponding decomposition G is monotone.*

Proof. The direct part is a special case of Lemma 2. For the converse part, observe that the ring A consisting of all functions in $C^*(X)$ which are stationary on each member of G is a closed subring of $C^*(X)$ by Lemma 1 as each $f \in A$ is the composite $g \cdot p$ for one and only one $g \in C^*(X/G)$, where p denotes the projection map of X onto X/G. The Stone-Weierstrass theorem then implies that A is the unique closed subring meeting the requirement of the proposition. Finally, that G is monotone if and only if A is analytic is immediate from [**1**, Theorem 16.30 and Lemma 16.31].

3. Examples of function rings on the n-sphere

The examples we present in this section will be subrings of $C^*(S^n)$ defined by means of certain monotone decompositions of S^n. Our most fundamental tool is the result that compact Hausdorff spaces X and Y are homemorphic if and only if $C^*(X)$ and $C^*(Y)$ are isomorphic. Beside this, we need the following result.

PROPOSITION 2. *Let K be a compact subset of a compact Hausdorff space X. Then the collection consisting of components of K and points of $X-K$ is an upper semicontinuous decomposition of X.*

This result is a consequence of the fact [**2**] that every component is a quasicomponent in a compact Hausdorff space. In fact, let C_1, C_2 be distinct components of K. Then K is the sum of two disjoint compact sets K_1 and K_2 with $C_1 \subset K_1$ and $C_2 \subset K_2$. Since K_1 and K_2 must be completely seperated, the decomposition has a Hausdorff quotient space.

THEOREM 1. *There is a monotone decreasing sequence $\{A_k\}$* of closed subrings of $C^*(S^n)$, $n \geq 2$, all isomorphic with $C^*(S^n)$ such that $\bigcap A_k$ is not isomorphic with $C^*(S^n)$ but contains a subring which is isomorphic with $C^*(S^n)$.

Proof. Without loss of generality, we assume that the complex number plane is a subspace of S^n by regarding S^n to be the one point compactification of

E^n. Let

$$H_1=\{x\in E^2\subset S^n : \|x\|=1 \ \& \ |\arg x|\leq\pi/3\},$$

and let

$$H_2=\{x\in E^2\subset S^n : \|x\|=1 \ \& \ |\arg x-\pi|\leq\pi/3\}.$$

For each positive integer $m>1$, let

$$H_{2m-1}=\{x\in E^2\subset S^n : \|x\|=1-2^{-m} \ \& \ |\arg x-\pi/2|\leq\pi/4\},$$

$$H_{2m}=\{x\in E^2\subset S^n : \|x\|=1-2^{-m} \ \& \ |\arg x-3\pi/2|\leq\pi/4\}.$$

For each $k\geq1$, let G_k be the decomposition of S_n into the sets $H_1, \cdots, H_k$ and points not lying on any of $H_1, \cdots, H_k$, and let A_k denote the ring consisting of all functions in $C^*(S^n)$ which are contant on each H_i, $i\leq k$. All G_k are upper semicontinuous by Proposition 2, and all S^n/G_k are homeomorphic to S^n by [3]. We have proved that all A_k are isomorphic with $C^*(S^n)$.

Every function in $\bigcap A_k$ is easily seen to be constant on the unit circle. Accordingly, each f in $\bigcap A_k$ is constant on the unit circle as well as on all H_k with $k\geq3$. On the other hand, the decomposition of S^n with unit circle and the sets H_k, $k\geq3$, as nondegenerate elements is upper semicontinuous by Proposition 2. Therefore, by Proposition 1, $\bigcap A_k$ consists of all functions in $C^*(S^n)$ which are constant on the unit circle as well as on each H_k, $k\geq3$. By [3], it suffices to show that the unit circle has complement distinct from E^n in order to prove that $\bigcap A_k$ fails to be isomorphic with $C^*(S^n)$. This follows, however, from [4, Corollary 4.8.14].

Finally, each A_k contains the subring consisting of functions in $C^*(S_n)$ which are constant on the unit disk in $E^2\subset S^n$. Since the corresponding decomposition has quotient space homeomorphic with S^n by virtue of [3], this subring is isomorphic with $C^*(S^n)$.

THEOREM 2. *There is a monotone decreasing sequence $\{A_k\}$ of mutually isomorphic subrings of $C^*(S^n)$, $n\geq2$, such that all A_k are not isomorphic with $C^*(S^n)$ but $\bigcap A_k$ is isomorphic with $C^*(S^n)$.*

Proof. For each positive integer k, let

$$H_k=\{x\in E^n\subset S^n : 2^{-k}\leq\|x\|\leq1\},$$

and let G_k be the decomposition of S^n with H_k as the only nondegenerate element, and let A_k be the ring consisting of all functions in $C^*(S^n)$ which are constant on H_k. Of course, G_k are upper semicontinuous by Proposition 2. Moreover, the rings A_k are all isomorphic because each S^n/G_k is the one point compactification of disjoint sum of two homeomorphic copies of E^n. However, non of the A_k is isomorphic with $C^*(S^n)$ because H_k is sent by the projection map to a cut point of the quotient space S^n/G_k.

Clearly, $\bigcap A_k$ has the unit n-ball as the only nondegenerate maximal stationary set, and it is clearly isomorphic with $C^*(S^n)$. This completes the proof of Theorem 2.

THEOREM 3. *There is a monotone increasing sequence* $\{A_k\}$ *of subrings of* $C^*(S^n)$, $n\geq 2$, *all isomorphic with* $C^*(S^n)$ *such that* $\bigcup A_k$ *is not isomorphic with* $C^*(S^n)$.

Proof. Let, for each positive integer k, H_k denote the set of points in $E^n \subset S^n$ with $\|x\| \leq 1/k$. Also, let G_k be the decomposition of S^n having H_k as the only nondegenerate element, and let A_k denote the ring consisting of all functions in $C^*(S^n)$ which are constant on H_k. All G_k are upper semicontinus by Proposition 2, and all S^n/G^k are homeomorphic to S^n. We have proved that all A_k are isormorphic with $C^*(S^n)$.

It only remains to prove that $\bigcup A_k$ fails to be isomorphic with $C^*(S^n)$. Now, suppose that $\bigcup A_k$ is isomorphic to $C^*(S^n)$. Since $\bigcup A_k$ must be closed by Lemma 1 in this case and each stationary set of $\bigcup A_k$ is a single point, we have $\bigcup A_k = C^*(S^n)$ by the Stone-Weierstrass theorem. This, however, leads to the contradiction that each function in $C^*(S^n)$ is constant on some neighborhood of the origin of $E^n \subset S^n$. The proof is completed.

5. Remarks on algebras over R

Let A be an algebra over a field F. We say that A has the monotone intersection property (MIP) on isomorphic subalgebras provided that every family of mutually isomorphic subalgebras of A totally ordered by inclusion has intersection isomorphic with the subalgebras belong to the family in question. Dually, A is said to have the monotone union property (MUP) on isomorphic subalgebras if monotone union of mutually isomorpic subalgebras of A is isomorphic with the summands. If A is finite dimensional over F, it certainly satisfies both MIP and MUP on isomorphic subalgebras.

Trivially, the real field $\mathbf{R}$ is an algebra over $\mathbf{R}$. The same is true for any (complete direct) product of copies of R under the addition, ring multiplication and multiplication by reals defined coordinatewise. In case A is infinite dimensional, MIP is of little interest as monotone intersection of isomorphic subalgebras may well have a zero ring as intersection. Instead, we are concerned with the weak monotone intersection property (WMIP) on isomorphic subalgebras: If L is a totally ordered family of subalgebras of A all containing, and isomorphic to, a subalgebra B of A, the intersection of members of L is isomorphic with B.

Our goal in this section is to point out that both WMIP and MUP on isomorphic subalgebras fail to hold in infinite products of $\mathbf{R}$. To do this, we need the following result which is a straightforward consequence of [**1**, Theorem 10.3].

LEMMA 3. *If X is a compact metric space, then $C^*(X)$ is isomorphic with a closed subring of $C^*(\mathbf{N})$.*

In the next theorem, "product" means complete direct product, and the product algebra $\mathbf{R}^{\mathbf{N}}$ is identical with the function algebra $C(\mathbf{N})$.

THEOREM 4. *Any infinite product of* $\mathbf{R}$ *has neither MUP nor WMIP on isomorphic subalgebras.*

Proof. Every infinite product of $\mathbf{R}$ contains a subalgebra isomorphic with the function algebra $C^*(\mathbf{N})$, and we may regard $C^*(S^n)$, $n\geq 2$, to be a subalgebra of the product algebra. Theorem 3 then disproves MUP, while Theorem 1 (or Theorem 2) denies WMIP on isomorphic subalgebra.

REMARK. The proof that MUP fails in an infinite product of reals rested on the fact that the union of the rings A_k in Theorem 3 is not closed. One might be tempted to assert that the union of an ascending sequence of isomorphic closed subrings of $C^*(S^n)$ has closure isomorphic with the rings in the sequence. This, however, is far from being true. In fact, let A_k, $k\geq 1$, denote the analytic subring whose nondegenerate maximal stationary sets are the circle with radius 2^{1-k} and the disk with radius 2^{-k}, both centered at the origin of $E^2\subset S^n$. Obviously, the A_k are mutually isomorphic but their union has closure $C^*(S^n)$, that is not isomorphic with A_k.

References

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