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ON SOME PROPERTIES OF φ-MIXING PROCESS

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1. Introduction

Mixing theory was introduced probability theory by A. Renyi, in 1958.

Subsequently, Bllingsley applied it in measure preserving transformation and F. Papangelou extended to Markov processes whose shift transformation is quasi-mixing (Lecture notes in mathematics 31. [5] pp. 272-279.)

From the view of the above mentioned works the theory is applicable in the study of stochastic processes to investigate the independency of random variable and the possibility to obtain the central limit theorem.

In this paper, I obtained some meaningful properties in the field of φ-mixing process by using Hölder and Minkowski's inequality, Markov chain and stationarity of process.

2. Backgound and definition

Let $\{X_n\}$, $n=0, \pm 1, \pm 2, \cdots$, be a strictly stationary sequence of random variables, defined on a probability space $(\Omega, \boldsymbol{B}, P)$, where Ω is a sample space, $\boldsymbol{B}$ is a σ-field of subsets of Ω and P is a probability measure on $\boldsymbol{B}$. For $a \leqq b$ define M_a^b as the σ-field generated by the random variables $\{X_n\}$. Similarly we can define $M_{-\infty}^{a}$ and M_a^∞ by the same way as above.

DEFINITION. If for each $k(-\infty < k < \infty)$, $n \geqq 1$,

$$E_1 \in M_{-\infty}^k,\ E_2 \in M_{k+n}^\infty,$$

$$|P(E_1 \cap E_2) - P(E_1) \cdot P(E_2)| \leqq \varphi(n) P(E_1) \tag{2-1}$$

then $\{X_n\} \in M_{-\infty}^\infty$ is φ-mixing process, where φ is a nonnegative function of positive integers.

Note. If $P(E_1) > 0$ then (2-1) is equivalent to

$$|P(E_2 | E_1) - P(E_2)| \leqq \varphi(n) \tag{2-2}$$

and therefore (2-2) implies that if $\varphi(n)$ is small, E_1 and E_2 are virtually independent.

In this paper I will use the following lemmas without proofs.

LEMMA I. *In Markov process, if the transition matrix* (P_{uv}) *is irreducible and aperiodic, then* $P_{uv}^{(n)} \to P_v$ (see [6] pp. 51–54).

LEMMA II. *If* f *and* g *are measurable function and if* $|f|^p$ *and* $|g|^q$ *are integrable, where* $p>1$ and $1/p+1/q=1$, *then* fg *is integrable, and for each measureable set* E,

$$\int_E |fg| \leqq \{\int_E |f|^p\}^{1/p}\{\int_E |q|^q\}^{1/q}$$

(see [4] pp. 238–239).

3. If the transition matrix (P_{uv}) is irreducible and aperiodic in Markov process, we can prove the following Theorem I.

THEOREM I. *Let* $\{X_n\}$ *be a stationary Markov process with finite state space. If the transition matrix* (P_{uv}) *is irreduceble and aperiodic, for*

$$E_1 = \{X_{k-i}, \cdots, X_k\} \in M_{-\infty}^{k}$$

$$E_2 = \{X_{k+n}, \cdots, X_{k+n+j}\} \in M_{k+n}^{\infty},$$

then we have

$$|P(E_1 \cap E_2) - P(E_1)\cdot P(E_2)| \leqq \varphi(n) P(E_1) \to 0 \tag{3-1}$$

Proof: Let P_u be a stationary probability in Markov process, and let P_{uv} be the transition probability. Then we have generally

$$P\{X_i = U_0, \cdots, X_{i+l} = U_l\} = P_{u_0}\ P_{u_0u_1} \cdots P_{u_{l-1}P_{u_l}} \tag{3-2}$$

for each finite sequence $u_0, u_1, \cdots, u_l$ of state.

Assume that the P_u are all positive, then we can easily know that

$$\varphi(n) = \max|P_{uv}/P_v - 1| \tag{3-3}$$

is finite.

Using (3-2) we have

$$|P(E_1 \cap E_2) - P(E_1)P(E_2)|$$

$$\leqq |P_{u_0}P_{u_0u_1} \cdots P_{u_{i-1}u_i} P_{v_iv_0}^{(n)} P_{v_0v_1} \cdots P_{v_{j-1}v_j} - P_{u_0}P_{u_0u_1} \cdots P_{u_{i-1}u_i} P_{v_0} P_{v_0v} \cdots P_{v_{j-1}v_j}|$$

$$= P_{u_0} \cdots P_{u_{i-1}u_i} |P_{u_iv_0}^{(n)} - P_{v_0}| P_{v_0v_1} \cdots P_{v_{j-1}v_j}$$

$$\leqq P_{u_0} \cdots P_{u_{i-1}u_i} |P_{u_iv_0}^{(n)} - P_{v_0}|$$

Thus, by Lemma I

$$\varphi(n) = \lim_{n\to\infty} \max|P_{u_iv_0}^{(n)} - P_{v_0}| = 0$$

then

$$|P(E_1 \cap E_2) - P(E_1)P(E_2)| \leqq \varphi(n) P(E_1) \to 0$$

4. Using the Hölder and Minkowski's inequality in φ-mixing process, we get the following result.

THEOREM II. *If* $\{X_n\}$ *is* φ*-mixing process, and* $M_{-\infty}^{k}$, M_{k+n}^{∞} *is the* σ*-field* *generated by* $\{X_n\}$ *for* $X\in M_{-\infty}^{k}$, $Y\in M_{k+n}^{\infty}$ $(n\geqq 0)$,

$$|E(XY)-E(X)E(Y)|\leqq 2\varphi^{1/p}(n)\{E|X|_p\}^{1/p}\{E|Y|_q\}^{1/q} \quad (4\text{-}1)$$

where

$$E|X|^p<\infty,\ E|Y|^q<\infty,\ 1/p+1/q=1$$

Proof: At first, we suppose two case $\varphi(n)=0$ and $\varphi(n)=1$.

(i) $\varphi(n)=0$. X and Y independent to each other, and thus the inequality (4-1) holds.

(ii) $\varphi(n)=1$. By Lemma II, the following holds evidently.

$$\begin{aligned}|E(XY)-E(X)E(Y)|&=|E[X(Y-E(Y))]|\\ &\leqq\{E|X|^p\}^{1/p}\{E|Y-E(Y)|^q\}^{1/q}\\ &\leqq 2\{E|X|^p\}^{1/p}\{E|Y|^q\}^{1/q}\end{aligned}$$

(iii) the general case. Suppose

$$X=\sum u_i I_{A_i},\quad Y=\sum v_j I_{B_j}$$

where $\{A_i\}$ and $\{B_j\}$ are finite decomposition of the sample space Ω into elements of $M_{-\infty}^{k}$ and M_{k+n}^{∞}.

$$\begin{aligned}|E(XY)-E(X)E(Y)|&=|\textstyle\sum_{ij}u_iv_j[P(A_i)P(B_j|A_i)-P(A_i)P(B_j)]|\\ &=|\textstyle\sum u_iP(A_i)\{\sum v_j(P(B_j|A_i)-P(B_j)\}|\\ &=|\textstyle\sum u_iP(A_i)^{1/p}[P(A_i)^{1/q}\sum v_j(P(B_j|A_i)-P(B_j)|\\ &\leqq\{\textstyle\sum|u_i^pP(A_i)|\}^{1/p}\{\sum P(A_i)|\sum v_j(P(B_j|A_i)\\ &\quad -P(B_j)^q\}^{1/q}\end{aligned}$$

But for each i,

$$\begin{aligned}|\textstyle\sum v_j(P(B_j|A_i)-P(B_i)|^q&=|\textstyle\sum v_i(P(B_j|A_i)-P(B_j)^{1/q}(P(B_j|A_i)-P(B_j))^{1/p}|^q\\ &\leqq\{\textstyle\sum|v_j^q(P(B_j|A_i)-P(B_j)|\}\{\sum|P(B_j|A_i)\\ &\quad -P(B_j)|\}^{q/p}\end{aligned}$$

and since

$$|\textstyle\sum P(A_i)\sum|v_j|^q||P(B_j|A_i)-P(B_j)|\leqq 2\ E|Y|^q$$

and so

$$\begin{aligned}&\{\textstyle\sum P(A_i)|\sum v_j(P(B_j|A_i)-P(B_j))|^q\}^{1/q}\\ &\quad\leqq\{\textstyle\sum P(A_i)\sum(v_j)^q|P(B_j|A_i)-P(B_j))|\}\{\sum|P(B_j|A_i)-P(B_j)|^{q/p}\}^{1/q}\\ &\quad\leqq 2^{1/q}\{E|Y|^q\}^{1/q}\{\textstyle\sum|P(B_j|A_i)-P(B_j)|\}^{1/q}\end{aligned}$$

If $C_i^+[C_i^-]$ is the union of B_j for which $P(B_j|A_i)-P(B_j)$ is positive (nonpositive), then C_i^+, C_j^- lise in M_{k+n}^{∞} and hence

$$\textstyle\sum|P(B_j|A_i)-P(B_j)|=P(C_i^+|A_i)-P(C_i^+)+P(C_i^-)-P(C_i^-|A_i)\leqq 2\varphi(n),$$

and then

$$\{\sum|P(B_j|A_i)-P(B_j)|\}^{1/p} \leqq 2^{1/p}\varphi(n)^{1/p}$$

therefore

$$|E(XY)-E(X)E(Y)| \leqq \{E|X|^p\}^{1/p}\, 2^{1/q}\{E|Y|^q\}^{1/q}\, 2^{1/p}\varphi(n)^{1/p}$$
$$=2\varphi(n)^{1/p}\{E|X|^p\}^{1/p}\{E|Y|^q\}^{1/q}$$

5. THEOREM III. *Let* $\{X_n\}$ *be stationary* φ*-mixing process and* $S_n=\sum_{i=1}^{n} X_i$. *if* $E(X_0)=0$, $E(X_0^2)<\infty$, *then*

$$n^{-1}E(S_n^2)\to E(X_0^2)+2\sum_{k=1}^{\infty} E(X_0X_k) \leqq 4E(X_0^2)\sum_{k=0}^{\infty}\varphi(k)^{1/2}$$

Proof. If $\rho_k=E(X_0X_k)$ then by stationarity,

$$E(S_n^2)=\sum_{k=1}^{n} E(X_0^2)+2\sum_{k\neq j} E(X_kX_j)$$
$$=n\rho_0+2\sum_{k=1}^{n-1}(n-k)\rho_k$$

therefore

$$n^{-1}E(S_n^2)=\rho_0+2\sum_{k=1}^{n-1}(1-k/n)\rho_k$$

$$\lim_{n\to\infty} n^{-1}E(S_n^2)=\rho_0+2\sum_{k=1}^{\infty}\rho_k=E(X_0^2)+2\sum_{k=1}^{\infty}E(X_0X_k)$$

by theorem II.

$$E(X_0^2) \leqq 2\varphi^{1/2}(0)E(X_0^2)$$
$$|E(X_0X_k)| \leqq 2\varphi^{1/2}(k)E(X_0^2)$$

therefore

$$\lim_{n\to\infty} n^{-1}E(S_n^2)=E(X_0^2)+2\sum_{k=1}^{\infty} E(X_0X_k)$$
$$\leqq 2\varphi^{1/2}(0)E(X_0^2)+4\sum_{k=1}^{\infty}\varphi^{1/2}(n)E(X_0^2)$$
$$\leqq 4E(X_0^2)\sum_{k=0}^{\infty}\varphi^{1/2}(k)$$

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