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SOME ALGEBRAIC PROPERTIES OF A TOPOLOGICAL SEMIFIELD $\mathbf{R}^A$

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The concept of topological semifields was first introduced by M. Antonovski, V. Boltyanski, and T. Sarimakov [1] in 1960. It is quite useful for the discussion of functional analysis. In this note we shall consider the special type of topological semifields which is a Tichonov product of reals and investigate some algebraic properties of this space.

Let A be an arbitrary set and let $\mathbf{R}^A$ be the set of all real valued functions on A. On $\mathbf{R}^A$ we define $+$ and $\cdot$ as follows: If $f, g \in \mathbf{R}^A$, $(f+g)(a)=f(a)+g(a)$ and $(f \cdot g)(a)=f(a)g(a)$ for all $a \in \mathbf{R}^A$. Let K be the set of all functions f in $\mathbf{R}^A$ such that $f(a)>0$ for every $a \in A$. We can topologize $\mathbf{R}^A$ by using axioms in [1] and obtain a topological semifield. It is easy to see that its semifield topology is equivalent to its product topology.

Let S be the set of all functions in $\mathbf{R}^A$ which are not equal to zero on at most a finite subset of A. Then S is a subring of the semifield $\mathbf{R}^A$, also an ideal of $\mathbf{R}^A$. This subring, however, is not a semifield. On the other hand the ring of integers is a subring of the semifield of reals, but not an ideal. The question then arises as to which subrings of the semifield $\mathbf{R}^A$ are ideals.

Let B be a subset of A. Then $\{f \in \mathbf{R}^A: f(a)=0 \text{ for every } a \in A \backslash B\}$ is isomorphic to $\mathbf{R}^B$. Moreover it is an ideal in $\mathbf{R}^A$. We are now ready to state our results.

THEOREM 1. *Let B be a subring of $\mathbf{R}^A$. Then B is an ideal of $\mathbf{R}^A$ iff fB is isomorphic to $\mathbf{R}^{A \setminus Z(f)}$, $Z(f)$ being the set of the zeros of f, for every nonzero function of B.*

Proof: Suppose that fB is isomorphic to $\mathbf{R}^{A \setminus Z(f)}$ for every nonzero $f \in B$. Since $\mathbf{R}^{A \setminus Z(f)}$ is isomorphic to a semifield ideal of $\mathbf{R}^A$ contained in B, $f \cdot g \in R^{A \setminus Z(f)} \subset B$ for any $g \in \mathbf{R}^A$. Conversely, suppose that B is an ideal in $\mathbf{R}^A$ and

$f\in B$. Consider $g\in\mathbf{R}^A$ such that $g(a)=0$ for $a\in Z(f)$ and $g(a)=1/f(a)$ for $a\in A\backslash Z(f)$. The function $f\cdot g$ belongs to B since B is an ideal. If $h\in\mathbf{R}^A$ and $Z(f)\subset Z(h)$, then $h=h\cdot f\cdot g$ and $h\in fB$. On the other hand, suppose that $h\in fB$. Then $h=f\cdot g$ for some $g\in B$, but then $h(x)=0$ for every x in $Z(f)$. Hence fB is isomorphic to $\mathbf{R}^{A\backslash Z(f)}$.

THEOREM 2. *Let B be a non-trivial ideal in $\mathbf{R}^A$. Then the following conditions are equivalent*: (1) *B is a principal ideal*, (2) *B has an identity*, (3) *B is closed.*

Proof: (1)$\rightarrow$(2): Suppose that B is generarted by f. Consider $g\in\mathbf{R}^A$ such that $g(a)=1/f(a)$ $a\in A\backslash Z(f)$ and $g(a)=0$ for $a\in Z(f)$. The function $f\cdot g$ is the characteristic function on $A\backslash Z(f)$ and belongs to B since B is an ideal.

(2)$\rightarrow$(1): Let i be an identity in B. Then clearly i generates B.

(3)$\rightarrow$(2): Let $T=\{a\in A\colon f(a)\neq 0$ for some $f\in B\}$. Then $\chi_{\{a\}}$, the characteristic function on $\{a\}$, belongs to B for every $a\in T$, furthermore χ_S belongs to B for every finite subset S of T. But then χ_T belongs to the closure of $\{\chi_S\colon S$ is a finite subset of $T\}$. Then, since B is closed, χ_T belongs to B.

(2)$\rightarrow$(3): We have shown that (1)$\leftrightarrow$(2). Hence $B=(\chi_B)$ and B is closed.

THEOREM 3. *Let F be a ring homomorphism of $\mathbf{R}^A$ into a topological ring B. If F is continuous, then* Ker F *is a topological semifield ideal.*

Proof: Ker $F=F^{-1}(0)$ is closed since F is continuous and Ker F is a semifield ideal by Theorem 2.

Referenes

[1] M. Antonovski, V. Boltyanski, T. Sarimakov, *Topological Semifield*, Izdat. Sarn. GU. Tashkent, 1960.

[2] K. Iseki, S. Kasahara, *On a Hahn-Banach type extension theorem*, Proc. Japan Acad., 41 (1965), 29-30

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