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## A SHUTTLE CAR REARRANGEMENT PROBLEM IN A COAL MINE

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### 1. Introduction

Suppose a coal mine composed of  $n$  sections and in each section two shuttle cars are assigned to carry coal from its underground coal face to the conveyer belt which links up to the coal train to surface. Restricting our attention only to operation of shuttle cars, we assume that the output (amount of coal per unit time) of each section is 1 when both shuttle cars are in operation,  $\delta$  ( $1 > \delta > 0$ ) when only one car is in operation, and 0 when no cars are in operation. The total output, as far as shuttle cars are concerned, depends on  $\delta$  and possible failures of either one car or two cars in different sections. And car failure depends on its machine design, age and history. Now, our shuttle car rearrangement problem may be stated as follows:

*Problem.* For  $2n$  shuttle cars in the mine when a simple ranking of those cars in increasing reliability is known, what is an optimum rearrangement of those cars which maximizes total expected output?

Reynolds [2] gave the following solution for the realistic case when  $\delta > 1/2$  and verified his solution, using a theorem on a system of common representatives in Combinatorial Mathematics.

*Solution.* Given a mine with  $n$  sections, let the shuttle cars  $c_1, c_2, \dots, c_n, c_{n+1}, \dots, c_{2n}$  be arranged by increasing order of reliability. Then the maximizing solution is given by the pairs  $\{c_1, c_{2n}\}, \{c_2, c_{2n-1}\}, \dots, \{c_n, c_{n+1}\}$ .

One of the purposes of this note is to give an elementary and self-contained proof of Reynolds' solution (Theorem 1). Secondly, we give a solution for the case  $\delta \leq 1/2$  (Theorem 2), and finally, we consider the case, for pure mathematical interest, of three shuttle cars in each section, instead of two, and give a partial solution (Theorem 3).

## 2. Mathematical model.

Let  $p_{k1}$  and  $p_{k2}$  be the probabilities of failure of the first and the second shuttle cars in the  $k$ th section. Then the probability of both cars in operation is given by  $(1-p_{k1})(1-p_{k2})$  (assuming that each car's operation is independent of one another). And the probability of only one car in operation in the  $k$ th section is  $p_{k1}(1-p_{k2}) + p_{k2}(1-p_{k1})$ . Therefore, the total expected output  $f$  is a function of a  $2 \times n$  matrix

$$S = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \end{pmatrix}$$

and thus we have

$$f(S) = \sum_{i=1}^n (1-p_{i1})(1-p_{i2}) + \delta \sum_{i=1}^n [p_{i1}(1-p_{i2}) + p_{i2}(1-p_{i1})]$$

or

$$(1) \quad f(S) = n + (\delta - 1) \sum_{i=1}^n (p_{i1} + p_{i2}) + (1 - 2\delta) \sum_{i=1}^n p_{i1} p_{i2}$$

In (1) the first sum will remain invariant but the second sum will be affected by a rearrangement. Hence to maximize  $f(S)$  we have to minimize  $\sum p_{i1} p_{i2}$  if  $\delta > 1/2$ , or to maximize the sum if  $\delta < 1/2$ . If  $\delta = 1/2$ ,  $f(S)$  will remain constant under any rearrangement. To derive the solution stated in the introduction from (1) for the case  $\delta > 1/2$  and to help other proofs the following simple lemma (see [1]) is essential.

LEMMA. For given two sets of positive reals:  $\{a_1, a_2, \dots, a_n\}$  and  $\{b_1, b_2, \dots, b_n\}$ ,

(i) the minimum of  $\sum_{i=1}^n a_i b_i$  corresponds to opposite ordering (e.g.,  $a_1 \leq a_2 \leq \dots \leq a_n$  and  $b_1 \geq b_2 \geq \dots \geq b_n$ ), and

(ii) the maximum of  $\sum_{i=1}^n a_i b_i$  corresponds to similar ordering of two sets (e.g.,  $a_1 \leq a_2 \leq \dots \leq a_n$  and  $b_1 \leq b_2 \leq \dots \leq b_n$ ).

*Proof.* If there are  $i$  and  $k$  such that  $a_i \leq a_k$  and  $b_i < b_k$ , then  $a_i b_i + a_k b_k - (a_i b_k + a_k b_i) = (a_i - a_k)(b_i - b_k) \geq 0$ .

## 3. Optimum rearrangement when $\delta > 1/2$ .

The following theorem clearly implies the above mentioned solution.

THEOREM 1. For  $2n$  positive reals  $a_1, a_2, \dots, a_{2n}$  the minimum of  $\sum_{i=1}^n a_i a_{2n+1-i}$

occurs when  $a_1 \leq a_2 \leq \dots \leq a_n \leq a_{n+1} \leq \dots \leq a_{2n}$ .

*Proof.* Suppose  $2n$  positive reals  $a_i$  are given with the above stated order. Let  $m$  be a real such that  $a_n \leq m \leq a_{n+1}$ . Now, suppose  $\sum_{i=1}^n x_i x'_i$  gives the minimum, where  $x_1, x_2, \dots, x_n, x'_1, \dots, x'_n$  is a rearrangement of  $a_i$ 's. If we assume without loss of generality  $x_1 \leq x_2 \leq \dots \leq x_n$  then by Lemma (i) we must have  $x'_1 \geq x'_2 \geq \dots \geq x'_n$ .

In a special case if  $x_1 \leq x_2 \leq \dots \leq x_n \leq m$  (or  $m \geq x'_1 \geq x'_2 \geq \dots \geq x'_n$ ) the matrix

$$\begin{pmatrix} x_1 & x_2 & \dots & x_n \\ x'_1 & x'_2 & \dots & x'_n \end{pmatrix} \left( \text{OR} \begin{pmatrix} x'_n & x'_{n-1} & \dots & x'_1 \\ x_n & x_{n-1} & \dots & x_1 \end{pmatrix} \right)$$

is identical with

$$\begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_{2n} & a_{2n-1} & \dots & a_{n+1} \end{pmatrix},$$

and hence the theorem holds.

In a general case if  $x_1 \leq x_2 \leq \dots \leq x_k \leq m < x_{k+1} \leq \dots \leq x_n$ , then it must be that  $m \geq x'_{k+1} \geq x'_{k+2} \geq \dots \geq x'_n$  and hence two sets  $\{x_1, x_2, \dots, x_k, x'_{k+1}, \dots, x'_n\}$  and  $\{a_1, a_2, \dots, a_n\}$  are equal. Since these numbers appear in every term of  $\sum_{i=1}^n x_i x'_i$  as only one factor, this sum can be rearranged and renamed as  $\sum_{i=1}^n a_i y_i$ . By Lemma (i) again,  $a_1 \leq a_2 \leq \dots \leq a_n$  implies  $y_1 \geq y_2 \geq \dots \geq y_n$ . Since also  $\{y_1, y_2, \dots, y_n\} = \{a_{2n}, a_{2n-1}, \dots, a_{n+1}\}$ , we have  $y_i = a_{2n+1-i}$  for  $1 \leq i \leq n$ . This completes the proof.

#### 4. Optimum rearrangement when $\delta < 1/2$ .

For this case we have to maximize  $\sum_{k=1}^n p_{k1} p_{k2}$  in (1) and the pairing in the solution should be rearranged as  $\{c_1, c_2\}, \{c_3, c_4\}, \dots, \{c_{2n-1}, c_{2n}\}$ . The next theorem verifies this assertion.

**THEOREM 2.** For given  $2n$  positive reals  $a_i$ 's the maximum rearrangement of

$$a_1 a_2 + a_3 a_4 + \dots + a_{2n-1} a_{2n}$$

occurs when  $a_1 \leq a_2 \leq a_3 \leq a_4 \leq \dots \leq a_{2n-1} \leq a_{2n}$ .

*Proof.* Suppose  $a_1 a_2 + a_3 a_4 + \dots + a_{2n-1} a_{2n}$  gives the maximum. There is no loss of generality in assuming  $a_1 \leq a_3 \leq \dots \leq a_{2n-1}$ . Then by Lemma (ii) it implies  $a_2 \leq a_4 \leq \dots \leq a_{2n}$ . Say  $a_1$  is the smallest. Then we can show  $a_2$  is the next smallest if  $a_2 \neq a_3$ . For, from our supposition  $a_1 a_2 + a_3 a_4$  must be also

the maximum for  $a_1, a_2, a_3$  and  $a_4$ , and then  $a_1a_2+a_3a_4=a_1a_2+a_4a_3$  and  $a_1\leq a_4$  imply  $a_2\leq a_3$  by Lemma (ii) for  $n=2$ . By the same argument if  $a_3$  is the third smallest then  $a_4$  is the forth, or if  $a_{2k+1}$  is the  $2k+1$ th smallest then  $a_{2k+2}$  is the next smallest one. This proves Theorem 2.

### 5. The case of three shuttle cars in each section.

First, we modify our previous assumptions for this case. Let the output of each section be 1 when all three cars are in operation,  $\delta$  when only two cars are in operation,  $\varepsilon$  when only one car is in operation, and 0 when no cars are in operation. Naturally  $1 > \delta > \varepsilon > 0$ . Then the total output is a function of

$$S = \begin{pmatrix} p_{11} & p_{21} & \cdots & p_{n1} \\ p_{12} & p_{22} & \cdots & p_{n2} \\ p_{13} & p_{23} & \cdots & p_{n3} \end{pmatrix}$$

where  $p_{ki}$  is the probability of failure of the  $i$ th car in the  $k$ th section, and

$$\begin{aligned} f(S) &= \sum_{k=1}^n (1-p_{k1})(1-p_{k2})(1-p_{k3}) \\ &\quad + \delta \sum_{k=1}^n [p_{k1}(1-p_{k2})(1-p_{k3}) + p_{k2}(1-p_{k1})(1-p_{k3}) + p_{k3}(1-p_{k1})(1-p_{k2})] \\ &\quad + \varepsilon \sum_{k=1}^n [p_{k1}p_{k2}(1-p_{k3}) + p_{k2}p_{k3}(1-p_{k1}) + (p_{k1}p_{k3}(1-p_{k2}))], \text{ or} \\ f(S) &= n + (\delta-1) \sum_i (p_{k1} + p_{k2} + p_{k3}) \\ &\quad + (1-2\delta+\varepsilon) \sum_i (p_{k1}p_{k2} + p_{k2}p_{k3} + p_{k3}p_{k1}) \\ &\quad + (3\delta-1-3\varepsilon) \sum_i p_{k1}p_{k2}p_{k3}. \end{aligned}$$

There are four cases to consider under the general restriction of  $1 < \varepsilon < \delta < 1$

*Case 1.*  $1+\varepsilon \geq 2\delta$  and  $3\delta \geq 1+3\varepsilon$ ,

*Case 2.*  $1+\varepsilon \geq 2\delta$  and  $3\delta < 1+3\varepsilon$ ,

*Case 3.*  $1+\varepsilon < 2\delta$  and  $3\delta \geq 1+3\varepsilon$ ,

*Case 4.*  $1+\varepsilon < 2\delta$  and  $3\delta < 1+3\varepsilon$ .

We are able to answer only for the case 1 by the following theorem.

**THEOREM 3.** For given  $3n$  positive reals  $a_1, b_1, c_1, a_2, b_2, c_2, \dots, a_n, b_n, c_n$ , the maxima of both (i)  $\sum_i a_i b_i c_i$  and (ii)  $\sum_i (a_i b_i + b_i c_i + c_i a_i)$  occur when e. g.,

$$a_1 \leq b_1 \leq c_1 \leq a_2 \leq b_2 \leq c_2 \leq \cdots \leq a_n \leq b_n \leq c_n$$

*Proof of (i).* Assume that  $\sum_{i=1}^n a_i b_i c_i$  gives the maximum and say  $a_1$  is

smallest number. Then clearly  $a_1b_1c_1+a_2b_2c_2$  has to be the maximum among those values obtained by permutations of six numbers involved. By Lemma (ii)  $a_1 \leq a_2$  implies  $b_1c_1 \leq b_2c_2$ . Hence

$$\begin{aligned} 0 &\leq a_1b_1c_1 + a_2b_2c_2 - a_1a_2c_1 - b_1b_2c_2 \\ &= (b_2c_2 - a_1c_1)(a_2 - b_1) \\ &= (b_2c_2 - b_1c_1 + b_1c_1 - a_1c_1)(a_2 - b_1) \end{aligned}$$

implies  $a_2 \geq b_1$ . Similarly,  $a_2 \geq c_1$ .

Thus we have shown that none of  $a_2, b_2$ , or  $c_2$  is less than  $b_1$  or  $c_1$  (there is no reason  $b_2$  or  $c_2$  are different from  $a_2$  in the above argument). By repeating the same argument we can show that  $a_{k+1}$  is no less than  $a_k, b_k$ , or  $c_k$  ( $k \geq 2$ ). This proves (i).

*Proof of (ii).* Assume  $\sum_i (a_i b_i + b_i c_i + c_i a_i)$  gives the maximum and  $a_1$  is a smallest number. Then  $\sum_{i=1}^2 (a_i b_i + b_i c_i + c_i a_i)$  is also the maximum among its rearrangements.

Now, since

$$\begin{aligned} 0 &\leq \sum_{i=1}^2 (a_i b_i + b_i c_i + c_i a_i) - (a_1 b_1 + b_1 c_2 + c_2 a_1 + a_2 b_2 + b_2 c_1 + c_1 a_2) \\ &= (c_2 - c_1)(a_2 - a_1 + b_2 - b_1), \end{aligned}$$

$b_1 \leq b_2$  implies  $c_1 \leq c_2$ . This means that if  $b_1$  is less than or equal to any one of  $a_2, b_2, c_2$  then we should have

$$a_1 \leq b_1 \leq c_1 \leq a_2 \leq b_2 \leq c_2$$

(with appropriate renaming between  $a_i, b_i$  and  $c_i$ ). Finally, it is sufficient to show that when  $a_1 < a_2$ , both  $b_1$  and  $c_1$  cannot be greater than each  $a_2, b_2$  and  $c_2$  at the same time. For, if they could then

$$\begin{aligned} a_1 b_2 + b_2 c_2 + c_2 a_1 + a_2 b_1 + b_1 c_1 + c_1 a_2 - \sum_{i=1}^2 (a_i b_i + b_i c_i + c_i a_i) \\ = (a_2 - a_1)(b_1 + c_1 - b_2 - c_2) > 0, \end{aligned}$$

and this means  $\sum_{i=1}^2$  is not the maximum, contrary to our assumption.

By the same argument we can assert that none of  $a_{k+1}, b_{k+1}$  and  $c_{k+1}$  are less than  $a_k, b_k$  or  $c_k$  ( $k \geq 2$ ). This completes proof of Theorem 3.

REMARKS. 1. It seems that an easy rearrangement pattern (such as in Theorems 1, 2 and 3) for minimum of  $\sum_i a_i b_i c_i$  or  $\sum_i (a_i b_i + b_i c_i + c_i a_i)$  does not exist in general. Even we find two rearrangements, say by computer,

to make the above two sums the minima, respectively, we do not necessarily have a solution for the case 4, unless these two rearrangements coincide.

2. To maximize  $f(S)$  for the cases 2, 3 and 4 we need to compute  $f(S)$  for all possible rearrangements. Hence we have to know the precise values of  $p_{ki}$ , in addition to the values of  $\delta$  and  $\varepsilon$ . Of course, a simple ranking of  $p_{ki}$  will give no light for these cases.

3. Theorem 3 can be generalized for  $kn$  positive reals ( $k \geq 3$ ).

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### References

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- [2] Garry H. Reynolds, *A Shuttle Car Assignment Problem in the Mining Industry*, Management Science, Vol. 17, No. 9, May, 1971, pp. 652-655.

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