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ON CONFORMAL KILLING TENSORS IN A RIEMANNIAN MANIFOLD

BY OKKYUNG YOON

1. Introduction

Let M^n be an n-dimensional Riemannian manifold with metric g_{ij}. Let ∇_i denote the operator of covariant differentiation with respect to the Riemannian connection. We denote the Riemannian curvature tensor, the Ricci tensor and the scalar curvature, respectively, by $R_{ijk}{}^h$, $R_{jk}=R_{ijk}{}^i$ and $R=g^{ij}R_{ij}$.

A skew symmetric tensor field u_{jk} is called a conformal Killing tensor if there exists a vector field q^h such that

$$\nabla_i u_{jk}+\nabla_j u_{ik}=2q_k g_{ij}-q_i g_{jk}-q_j g_{ik}.$$

Such a vector field q^h is called an associated vector of u_{jk} and is given by

$$\nabla^l u_{lk}=(n-1)q_k.$$

In a Kählerian manifold $\mathfrak{M}^n$ with a mixed tensor $F_i{}^h$ and with a Riemannian metric g_{ij}, a skew symmetric tensor field w_{jk} is called a F-conformal Killing tensor, if there exist two vector fields p^i and q^i such that

$$\nabla_i w_{jk}+\nabla_j w_{ik}=2q_k g_{ij}-q_i g_{jk}-q_j g_{ik}+3(p_i F_{jk}+p_j F_{ik}).$$

S. Tachibana [1] and C. H. Chen [2] studied such a tensor.

In this paper, we shall study another form of conformal Killing tensor in a conformally flat space.

2. Pseudo F-conformal Killing tensor

For a skew symmetric tensor field f_{jk} in M^n, if there exists a covector field p_i such that

$$\nabla_j f_k{}^h=q^h g_{jk}-q_k\delta_j^h-p^h f_{jk}+p_k f_j{}^h-\alpha p_j f_k{}^h, \tag{2.1}$$

where

$$p_i=\partial_i p,\quad q_k=p_l f_k{}^l,\quad p^h=p_l g^{lh},\quad q^h=q_l g^{lh},\quad f_k{}^h=f_{kl}g^{lh}$$

and α is an arbitrary constant, then we have

$$\nabla_i f_{jk}+\nabla_j f_{ik}=2q_k g_{ji}-q_j g_{ik}-q_i g_{jk}+(1-\alpha)(p_i f_{jk}+p_j f_{ik}). \tag{2.2}$$

If $\alpha=1$, then f_{jk} is a conformal Killing tensor, and $\alpha\neq 1$, then f_{jk} is analogous to an F-conformal Killing tensor w_{jk} in $\mathfrak{M}^n$. Such a tensor f_{jk} defined as (2.1), we shall call a *pseudo F-conformal Killing tensor*, p_k is associated vector of f_{jk}, and α is *index number* of f_{jk}.

THEOREM 1. (*Condition of Integrability*) *The necessaey and sufficient condition that there exists a pseudo F-conformal Killing tensor in an n-dimensional Riemannian manifold* ($n\geqq 3$), *is that the given manifold be conformally flat.*

Proof. Differentiating covariantly (2.1) and by making use of

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$$\nabla_i\nabla_j f_k{}^h - \nabla_j\nabla_i f_k{}^h = f_k{}^l R_{ijl}{}^h - f_l{}^h R_{ijk}{}^l$$

we have

(2.3)
$$f_k{}^l R_{ijl}{}^h - f_l{}^h R_{ijk}{}^l = g_{jk} q_i{}^h - g_{ik} q_j{}^h - \delta_j{}^h q_{ik} + \delta_i{}^h q_{jk} + f_j{}^h p_{ik} - f_i{}^h p_{jk} - f_{jk} p_i{}^h + f_{ik} p_j{}^h - \alpha f_k{}^h \ (p_{ij} - p_{ji}),$$

where

$$p_{jk} = \nabla_j p_k - p_j p_k + \frac{1}{2}(p_l p^l) g_{jk}, \quad p_j{}^h = p_{jl} g^{lh},$$

$$q_{jk} = \nabla_j q_k - q_j p_k + \alpha p_j q_k + \frac{1}{2}(p_l p^l) f_{jk}, \quad q_j{}^h = q_{jl} g^{lh}.$$

On the other hand, from $q_k = p_l f_k{}^l$, by calculating we have

(2.4)
$$q_{jk} = p_{jl} f_k{}^l$$

and by assumption $p_{ij} = p_{ji}$. If we put

$$W_{ijk}{}^h = R_{ijk}{}^h - g_{jk} p_i{}^h + g_{ik} p_j{}^h + \delta_j{}^h p_{ik} - \delta_i{}^h p_{jk},$$

and substituting (2.4) into (2.3), then we have

$$f_k{}^l W_{ijl}{}^h = f_l{}^h W_{ijk}{}^l.$$

Since the last equation is valid for any f_{jk}, we have

$$W_{ijk}{}^h = 0$$

that is,

(2.5)
$$R_{ijk}{}^h = g_{jk} p_i{}^h + g_{ik} p_j{}^h + \delta_j{}^h p_{ik} - \delta_i{}^h p_{jk}.$$

Contracting for h and i, we have

$$R_{jk} = g_{jk} p_l{}^l + (n-2) p_{jk}$$

and transvecting with g^{jk}, we have

$$R = 2(n-1) p_l{}^l.$$

Therefore, we get

$$p_{jk} = \frac{1}{n-2} R_{jk} - \frac{R}{2(n-1)(n-2)} g_{jk}.$$

By virtue of the last equation, (2.5) becomes

(2.6)
$$R_{ijk}{}^h + \frac{1}{n-2}(\delta_j{}^h R_{ik} - \delta_i{}^h R_{jk} + g_{ik} R_j{}^h - g_{jk} R_i{}^h) + \frac{R}{(n-1)(n-2)}(\delta_i{}^h g_{jk} - \delta_j{}^h g_{ik}) = 0$$

Since the left hand side in (2.6) is a Wyle's conformal curvature tensor, the manifold is conformally flat. Thus the proof is completed.

3. An almost complex manifold admitting pseudo F-conformal Killing tensor

Let $C^n (n \geqq 4)$ be an almost complex manifold with a structure tensor $F_i{}^h$, Riemannian metric g_{ij}, and a symmetric connection satisfying the following conditions

$$F_i{}^l F_l{}^h = -\delta_i{}^h, \quad F_{ij} = F_i{}^l g_{lj} = -F_{ji}.$$

If the almost complex structure tensor $F_i{}^h$ is a pseudo F-comformal Killing tensor, then we have index number $\alpha=0$, from which

$$\nabla_i(F_j{}^l F_l{}^h)=F_l{}^h\nabla_i F_j{}^l+F_j{}^l\nabla_i F_l{}^h=0.$$

In this case, $F_i{}^h$ is given by

$$\nabla_j F_k{}^h=q^h g_{jk}-q_k\delta_j{}^h-p^h F_{jk}+p_k F_j{}^h. \tag{3.1}$$

By calculating, we can easily find that Nijenhuis tensor $N_{jk}{}^h$ of $F_i{}^h$ vanishes. Therefore $F_i{}^h$ becomes ccmplex structure tensor.

If we consider a conformal change of the affine connection satisying

$$\bar{\Gamma}_{jk}{}^h=\mu_{jk}{}^h+p_j\delta_k{}^h+p_k\delta_j{}^h-p^h g_{jk},$$

then covariant devivative $\bar{\nabla}_j F_k{}^h$ with respect to $\bar{\Gamma}_{jk}{}^h$ vanishes.

From the above discussion and Theorem 1, we have

THEOREM 2. *If an almost complex manifold with structure (g, F) is a conformally flat space, then there exists a complex structure tensor $F_i{}^h$ which is a pseudo F-conformal Killing tensor defined by the differential equations*

$$\nabla_j F_k{}^h=q^h g_{jk}-q_k\delta_j{}^h-p^h F_{jk}+p_k F_j{}^h,$$

where $p_i=\partial_i p$, $q_k=p_l F_k{}^l$. Moreover, the covariant derivative of $F_i{}^h$ with respect to the conformal connection of $\Gamma_{jk}{}^h$ defined by

$$\bar{\Gamma}_{jk}{}^h=\Gamma_{jk}{}^h+p_j\delta_k{}^h+p_k\delta_j{}^h-p^h g_{jk}$$

vanishes.

References

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Seoul National University