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ON ORDERS IN A CLASS OF *QF*-3 RINGS

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1. Introduction

Throughout this paper, we assume that every ring has an identity $1\neq 0$, and every module is unitary unless mentioned otherwise. Let M_R (resp. $_RM$) denote the right (resp. left) R-module, and $E_R(M)$ the injective hull [3] of M. An over-ring Q of R is called a (right) classical quotient ring of R if and only if (1) every regular element of R has a two-sided inverse in Q, (2) every element of Q has the form ab^{-1} where a in R, b $(\neq 0)$ regular in R. In this case the subring R is called a (right) order in Q. Left classical quotient ring and left order are defined in a similar fashion.

A ring R is called right QF-3 provided that R has a minimal faithful right R-module [5, p. 1105], i.e., a faithful right R-module which is isomorphic to a direct summand of every faithful right R-module.

It is well known that if R is right Artinian, the following are equivalent [5]:

(1) R has a minimal faithful module M.
(2) R has a faithful, projective, injective right ideal (which is isomorphic to M).
(3) $E_R(R)$ is projective.

But recently the above QF-3 rings were classified by H. Tachikawa [9, p. 225] as follows:

(1) R is called right QF-3 if R has a direct summand eR (e is an idempotent of R) which is a faithful, injective right ideal.
(2) R is called right QF-3^+ if the injective hull $E(R)$ is projective.
(3) R is called right QF-$3'$ if the injective hull $E(R)$ of R_R is torsionless in the sense of H. Bass [1, p. 476] and T. Kato [7, Section 1].

The class of right QF-$3'$ rings is the most general of the above [9], and the notions of QF-3^+ and QF-$3'$ rings are first suggested by Wu, Mochizuki and Jans [12] Kato [7].

C. Vinsonhaler [11, Theorem 6] has investigated orders in Artinian QF-3 rings. The main results of this paper are concerned with characterizations of orders in the three classes of QF-3 rings, and relations between orders and QF-3 over-rings. In Section 2, the necessary and sufficient conditions of rings whose quotient rings to be QF-$3'$ and QF-3^+ rings will be investigated, in Section 3 the necessary and sufficient conditions that a ring to be embedded in a QF-3 ring as a right order and in a QF-$3'$ ring as a right order will be investigated.

2. *QF*-3 quotient rings

Let M be an R-module and N be a submodule of M. M is called a rational extension [4, p. 58] of N in case $f(N)=0$ implies $f=0$ for f in $\mathrm{Hom}_R(L, M)$, where L is any submodule of M containing N. An over-ring Q of R is said to be a right quotient ring of R if Q_R is a rational extension of R_R. In [10] Utumi has defined an extension ring

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Q of R to be a right quotient ring of R if and only if for any $a, b(\neq 0)$ in Q, there is r in R such that ar is in R and $br\neq 0$. But Faith [4, p. 58] has shown that the above two definitions about quotient ring are equivalent.

LEMMA 1. *Let Q be a right quotient ring of R and M be a right R-module. Then* $\mathrm{Hom}_R(M, Q)=\mathrm{Hom}_Q(M, Q)$.

Proof. Clearly, $\mathrm{Hom}_R(M, Q)$ contains $\mathrm{Hom}_Q(M, Q)$. Conversely, for any f in $\mathrm{Hom}_R(M, Q)$, let $\varphi(q)=f(m_0q)-f(m_0)q$ for some m_0 in M and for any q in Q. Then it is a routine verification to show that $\varphi\in\mathrm{Hom}_R(Q, Q)$ and $\varphi(R)=0$. By assumption, since Q_R is a rational extension of R_R, $\varphi=0$, i.e., this means $f(m_0q)-f(m_0)q=0$ for all $q\in Q$. Therefore f is a Q-homomorphism from M to Q. Thus the proof is completed.

H. Tachikawa [9, Proposition 1.1] has shown that each class of QF-3 rings classified is closed under taking quotient rings, that is, a right quotient ring of QF-3, QF-3$^+$ and QF-3$'$ ring is QE-3, QF-3$^+$ and QF-3$'$ ring respectively. But does the converses hold? If not, possibly under what conditions? A partial answer to this question is given as follows.

PROPOSITION 2. *Let Q be a right quotient ring of R. Then Q is Q F-3$'$ if and only if for any r in R, there exists f in* $\mathrm{Hom}_R(E(R), Q)$ *such that* $f(r)\neq 0$.

Proof. We shall complete the sufficiency by showing that $E(Q)$ is Q-torsionless, i.e., for any $q(\neq 0)$ in $E(Q)$, there exists f in $\mathrm{Hom}_Q(E(Q), Q)$ such that $f(q)\neq 0$. We pick out $q(\neq 0)$ in $E(Q)$, then qR has a non-zero intersection with R by the equality of $E(Q)$ and $E(R)$. Then we can select a non-zero element $r_0=qr\in R\cap qR$. By assumption, there exists f in $\mathrm{Hom}_R(E(R), Q)$ such that $0\neq f(r_0)=f(qr)=f(q)r$. And by Lemma 1, this f can be considered as a Q-homomorphism from $E(Q)$ to Q. Conversely, suppose Q is QF-3$'$. Then $E(Q)$ is Q-torsionless. Thus the necessity is satisfied by Lemma 1.

REMARK 3. The condition

(A) for any r in R, there exists f in $\mathrm{Hom}_R(E(R), Q)$ such that $f(r)\neq 0$

is strictly weaker than the following condition, (for example the integer ring Z)

(B) R is QF-3$'$, i.e., $E(R)$ is R-torsionless.

The above Proposition 2 shows that condition (A) implies QF-3$'$ ness of Q. This is a kind of converse of Tachikawa's [9, Proposition 1.1].

Dual Basis Lemma. [2, VII, Proposition 3.1] *A right R-module P is projective if and only if there exists subsets $\{a_i\}$ of P and $\{f_i\}$ of* $\mathrm{Hom}_R(P, R)$ *such that for each $a\in P, f_i(a)=0$ for almost all i, and $a=\sum a_if_i(a)$.*

From this Lemma, we obtain the following.

PROPOSITION 4. *A necessary and sufficient condition that a quotient ring Q of R to be QF-3$^+$ and $E(Q)$ is Q-torsionfree is that R satisfies the following conditions;*

(1) *there exists subsets $\{a_i\}$ of $E(R)$ and $\{f_i\}$ of* $\mathrm{Hom}_R(E(R), Q)$ *such that for each $c\in R$, $f_i(c)=0$ for almost all i, and $c=\sum a_if_i(c)$.*

(2) *R has a R-torsionfree injective hull.*

Proof. Straightforward.

3. Orders in QF-3 rings

To begin with, the following definitions shall be introduced.

DEFINITION 5. [6, Definition 4] If M is an R-module, let

$$T(M)=\{m\in M \mid \text{there exists } b \text{ regular in } R \text{ such that } mb=0\}.$$

DEFINITION 6. [11, Definition, p. 84] An R-module M is called *regular divisible* provided $T(M)=0$, and the equation $xb=m$ has a solution x for all m in M, b regular in R.

We shall use the method of Vinsonhaler [11] whish is an extension of the method by Jans [6] for developing quotient rings. Let φ be the collection of all R-modules M such that $R\subseteq M\subseteq E(R)$, and $T(M/R)=M/R$. Then φ contains R_R and hence is non-empty. Since $(\varphi, \subseteq)$ is inductive, there is a maximal element in φ by Zorn's Lemma. In this case the sum of any two elements of φ is again in φ, so the maximal element is unique and C. Vinsonhaler [11] has denoted this by $U(R)$.

The following theorem is a generalization of Park's result [8, Theorem 3] and a modification of Vinsonhaler's result [11, Theorem 5].

THEOREM 7. *We can give a ring structure on $U(R)$ having R as a subring.*

Proof. First we have $\mathrm{Hom}_R(U(R)/R, U(R))=0$. Suppose that $\mathrm{Hom}_R(U(R)/R, U(R))\neq 0$, then we can choose f in $\mathrm{Hom}_R(U(R)/R, U(R))$, $\mathrm{Im}(f)\neq 0$. Since $R\subseteq U(R)\subseteq E(R)$ and $E(R)$ is an essential extension of R_R, there is c in $U(R)$ such that $a=f(\bar{c})\in \mathrm{Im}(f)\cap R$, where $\bar{c}=c+R\in U(R)/R$. Then we have a regular $b(\neq 0)$ in R, $\bar{c}b=cb+R=0$ by definition of $U(R)$. In this case, $0=f(\bar{c}b)=f(\bar{c})b=ab$, a contradiction. Now the exact sequence

$$0\longrightarrow R\longrightarrow U(R)\longrightarrow U(R)/R\longrightarrow 0$$

induces the following exact sequence

$$0\longrightarrow \mathrm{Hom}_R(U, U)\longrightarrow \mathrm{Hom}_R(R, U)\longrightarrow 0$$

And applying Park's result [8, Theorem 3], we can give a ring structure on $U(R)$ having R as a subring.

REMARK 8. *By the above theorem, $U(R)$ can be made into a (right) classical quotient ring of R if $U(R)$ is regular divisible R-module.*

THEOREM 9. *A necessary and sufficient condition that a ring R can be imbedded in a QF-3 ring as a (right) order is that R satisfies the following conditions;*

(1) *$U(R)$ is regular divisible.*

(2) *there exists an R-injective faithful submodule of $U(R)$.*

Proof. Suppose R satisfies the given conditions (1) and (2). Then $U(R)$ is a (right) classical quotient ring of R by Theorem 7 and Remark 8. And since an R-injective faithful submodule of $U(R)$ is also $U(R)$-injective faithful by the process of proof in a Tachikawa's result [9, Proposition 1.2], $U(R)$ is a QF-3 ring which has a subring R as a (right) order. Conversely, let R be a (right) order in a QF-3 ring Q. Then it is easily verified that $U(R)=Q$ by the maximality of $U(R)$. Since Q is a (right) classical quotient ring of R, $Q=U(R)$ satisfies the condition (1). And $Q=U(R)$ satisfies (2) by the assumption. Thus the proof is completed.

The above theorem is a modification of the result of Vinsonhaler [**11**, Theorem 6] without assumption of minimum condition.

THEOREM 10. *A necessary and sufficient condition that a ring R to be a (right) order in a QF-$3'$ ring is that R satisfies the following conditions;*

(1) *$U(R)$ is regular divisible.*

(2) *for any r in R, there is an element f in* Hom_R $(E(R), U(R))$ *such that* $f(r)\neq 0$.

Proof. The above theorem follows from Proposition 2 and Theorem 9.

Similarly we get easily the following theorem from Proposition 4 and Theorem 10.

THEOREM 11. *A necessary and sufficient condition that a (right) classical quotient ring Q of R to be QF-$3'$ and $E(Q)$ is Q-torsionfree is that R satisfies the following conditions;*

(1) *there exists subsets $\{a_i\}$ of $E(R)$ and $\{f_i\}$ of* Hom_R $(E(R), Q)$ *such that for each c in R, $f_i(c)=0$ for almost all i, and $c=\sum a_i f_i(c)$.*

(2) *R has a torsionfree injective hull.*

(3) *$U(R)$ is regular divisible.*

REMARK 12. The authors were not able to establish a condition that a ring R to be imbedded as a right order in a QF-3^+ ring Q.

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