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ON THE CATEGORY OF VECTOR LATTICES

BY DONG SUN SHIN

1. Introduction

Let E be a vector lattice and M be a lattice ideal of E. Then we obtain the quotient vector lattice E/M and a lattice homomorphism $p: E \to E/M$. The purpose of the present paper is to show that

(a) Every lattice homomorphism on a vector lattice E with kernel M has an image isomorphic to E/M, and

(b) $p: E \to E/M$ is a universal element for a suitable functor (cf. Theorem 2).

For the terminologies used in the present paper we refer to the papers [3] and [4].

2. Preliminaries

An ordered vector space is a real vector space E equipped with a transitive, reflexive, antisymmetric relation satisfying the following conditions:

(1) If x, y, z are elements of E and $x \leqq y$, then

$$x+z \leqq y+z$$

(2) If x and y are elements of E and α is a positive real number, then $x \leqq y$ implies $\alpha x \leqq \alpha y$.

The positive cone (or simply the cone) K in an ordered vector space E is defined by $K = \{x \mid x \subset E,\ x \geqq 0\}$, where 0 denote the zero element of E. The cone K has the following "geometric" properties:

(c_1) $K+K \subset K$,

(c_2) $\alpha K \subset K$ for each real number $\alpha > 0$, and

(c_3) $K \cap (-K) = \{0\}$.

If K is a subset of a real vector space E satisfying (c_1), (c_2) and (c_3), then $x \leqq y$ if $y - x \in K$ defines an order relation on E with respect to which E is an ordered vector space with positive cone K.

A subset K of E containing zero element and satisfying (c_1) and (c_2) is called a wedge.

DEFINITION 1. An ordered vector space $(E, \leqq)$ is called a *vector lattice* if and only if for each x and y in E there is a unique supremum of x and y in E.

DEFINITION 2. A linear subspace M of a vector lattice E is called a *lattice ideal* if $y \in M$ whenever $x \in M$ and $|y| \leqq |x|$.

DEFINITION 3. A linear mapping f on a vector lattice E to another vector lattice F is a *lattice homomorphism* if and only if $f(x \wedge y) = f(x) \wedge f(y)$ for all x and y in E.

A one-one (into) lattice homomorphism is called a *lattice isomorphism.*

If M is a linear subspace of a vector space E ordered by a cone K, the image $C = p(K)$ of K under the canonical quotient mapping $p: E \to E/M = F$ is a wedge in E. However C is not a cone in general, even if E is a vector lattice and M is a sublattice of E. The next result shows that a much better order theoretic correspondence between E

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and F is valid if M is a lattice ideal.

PROPOSITION 1. *If E is a vector lattice and M is a lattice ideal in E, the quotient space $F=E/M$ is a vector lattice for the order structure determined by the canonical image C in F of the cone K.*

Proof. Refer to [4], p. 37.

By the proposition 1 we obtain a vector lattice E/M. This vector lattice E/M is called the quotient vector lattice of E by its lattice ideal. The canonical mapping p: $E\to E/M$ may be described as the function assigning to each $x\in E$ the unique $M+x$. In the following proposition 2 the canonical mapping $p:E\to E/M$ of vector lattice E onto the quotient vector lattice E/M is a lattice homomorphism.

PROPOSITION 2. *For a vector lattice E/M the canonical mapping $p:E\to E/M$ is a lattice homomorphism, where M is a lattice ideal in E.*

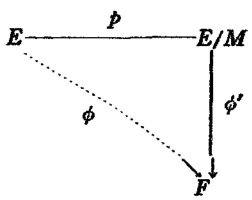
Proof. Refer to [1], p. 525.

3. Theorems

The vector lattice E/M can be characterized by the following theorem of the lattice homomorphism $p:E\to E/M$:

THEOREM 1. *For each lattice homomorphism $\phi:E\to F$ of vector lattices with $\phi(M)=0$, where M is a lattice ideal in a vector lattice E, there is a unique lattice homomorphism $\phi':E/M\to F$ such that $\phi=\phi'\circ p$.*

Proof. The situation in the theorem is as indicated in the diagram below:



Given the lattice homomorphisms ϕ and p, it suffices to find a lattice homomorphism ϕ' in such a way that the diagram commutes.

If we define $\phi'(M+a)=\phi(a)$, then ϕ' is well defined. In fact, if $M+a=M+b$ $(a,b\in E)$, then $M+(a-b)=0$, which implies that $a-b\in M$. Hence $\phi(a-b)=0$. Since ϕ is linear, $\phi(a)=\phi(b)$. Therefore, $\phi'(M+a)=\phi'(M+b)$.

ϕ' carries all the elements of $M+a$ to a single element $\phi(a)$ of F. This means that there is a unique function ϕ' on E/M to F with $\phi'\circ p=\phi$.

This function ϕ' is a linear mapping, since for any $M+a$, $M+b$ in E/M $(a,b\in E)$ and for any scalar λ,

$$\begin{aligned}\phi'[(M+a)+(M+b)]&=\phi'[M+(a+b)]\\&=\phi(a+b)\\&=\phi(a)+\phi(b)\\&=\phi'(M+a)+\phi'(M+b),\end{aligned}$$

$$\phi'[\lambda(M+a)]=\phi'(M+\lambda a)=\phi(\lambda a)=\lambda\phi(a)=\lambda\phi'(M+a).$$

To show that ϕ' is a lattice homomorphism, we need only show that for any $M+b$

and $M+a$ in E/M $(a, b \in E)$, we have

$$\phi'[(M+a)\vee(M+b)]=\phi'(M+a)\vee\phi'(M+b).$$

Since p and ϕ are lattice homomorphisms,

$$\begin{aligned}(M+a)\vee(M+b) &= p(a)\vee p(b)\\ &= p(a\vee b)\\ &= M+(a\vee b).\end{aligned}$$

Hence $\phi'[(M+a)\vee(M+b)]=\phi'[M+(a\vee b)]=\phi(a\vee b)=\phi(a)\vee\phi(b)=\phi'(M+a)+\phi'(M+b)$. This completes the proof.

For each lattice homomorphism $\phi: E\to F$ the kernel of ϕ is a lattice ideal of E (cf., [1], p. 525). Therefore we have the following corollary:

COROLLARY. *For the above lattice homomorphism ϕ': $E/M\to F$, ϕ' is a lattice isomorphism if ϕ: $E\to F$ is a lattice homomorphism of vector lattice with kernel M.*

Proof. To show that ϕ' is a lattice isomorphism we need only show that ϕ' is one-one. If $\phi'(M+a)=\phi'(M+b)$ for any $M+a, M+b$ in E/M, then $\phi(a)=\phi(b)$, which means that $a-b\in M$. Therefore $M+a=M+b$. Hence ϕ' is a lattice isomorphism.

We now construct the category of vector lattices. Let the object be a vector lattice and let "hom" be the function with hom $(E, F)=\{\phi|\phi: E\to F$ is a lattice homomorphism$\}$. Since each identity 1_E: $E\to E$ is a lattice homomorphism of vector lattice E, and since the composite of two lattice homomorphisms of vector lattices is again a lattice homomorphism, these data do determine a category of vector lattices.

Let X be the category of vector lattices and let Y be the category of sets. For each vector lattice F and a fixed vector lattice E, we define $\mathfrak{F}$ as follows:

$$\mathfrak{F}(F)=\text{hom}(E, F)=\{\phi|\phi: E\to F \text{ is a lattice hom.}\};$$

and for each lattice homomorphism $f: S\to T (S, T\in X)$,

$$\mathfrak{F}(f): \mathfrak{F}(S)\to\mathfrak{F}(T)$$

$$\mathfrak{F}(f)\alpha=f\circ\alpha \qquad \text{for any } \alpha\in\mathfrak{F}(S).$$

Then we have a covariant functor $\mathfrak{F}$ from X into Y.

For a lattice ideal M of a fixed vector lattice E, we define $\mathfrak{F}_M$ as follows:

$$\mathfrak{F}_M(F)=\{\phi|\phi: E\to F \text{ is a lattice hom., and } \phi(M)=0\};$$

and for each lattice homomorphism $f: S\to T$ $(S, T\in X)$,

$$\mathfrak{F}_M(f)=\mathfrak{F}(f).$$

Then $\mathfrak{F}_M$ is a subfunctor of $\mathfrak{F}$.

Therefore we obtain the following result from the above discussions and Theorem 1.

THEOREM 2. *The subfunctor $\mathfrak{F}_M$ of the covariant functor $\mathfrak{F}$: $X\to Y$ has a universal element $(p, E/M)$.*

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Ewha Woman's University