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ON THE RIGHT QUOTIENT MONOIDS IN $U[y, D]$

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1. Introduction.

In [3] a field U with derivation D is called universal differential field if the equation

$$p(x,\ D(x), \cdots, D^{(n)}(x))=0, \quad n\text{:arbitrary},$$

has a solution u in U. Furthermore, every homogeneous linear differential equation in D over U has a nontrivial solution in U. Let $U[y, D]$ denote the ring of differential polynomials in the indeterminate y with coefficients in U, and multiplication in $U[y, D]$ is defined by: $ya=ay+D(a)$ for all a in U. Then $U[y, D]$ is a principal right ideal domain.

Recently, R. A. Beauregard has concerned the localization in a principal right ideal domain R using properties of right quotient monoid S in R and right quotient ring of R with respect to S [1].

The main purpose of this note is to find the right quotient monoids S in $U[y, D]$ and right quotient rings of $U[y, D]$ with respect to S.

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2. Right Quotient Monoids in $U[y, D]$

DEFINITION 1. The *degree* $\deg(f)$ of $f=\sum_{i=0}^{n} a_i y^i$ in $U[y, D]$ is the largest integer n such that $a_n \neq 0$.

DEFINITION 2. A subset $S \neq \phi$ of nonzero elements of an integral domain R (not necessarily commutative) is *a right quotient monoid* in R if

(1) $ab \in S$ iff $a, b \in S$ where $a, b \in R$,

(2) $a \in S, b \in R$ implies $aR \cap bS \neq \phi$.

DEFINITION 3. For a, b in an integral domain R a and b are *similar* if R/aR is isomorphic to R/bR as right R-modules.

PROPOSITION 1. *In* $U[y, D]$, $y-a$ *and* $y-b$ *are similar for all* $a, b \in U$.

Proof. If $c(y-a)=(y-b)d$ for some nonzero $c, d \in U$, then $c=d$ and $(a-b)c+D(c)=0$. Since U is universal differential field, $(a-b)x+D(x)=0$ has a nontrivial solution c in U. We shall show the map β of $U[y, D]/(y-a)U[y, D]$ into $U[y, D]/(y-b)U[y, D]$ defined by $\beta(f+(y-a)U[y, D])=cf+(y-b)U[y, D]$ where nonzero c in U such that $c(y-a)=(y-b)c$, is an isomorphism. We know the map β is a homomorphism. It suffices to show that β is one-to-one and onto. Since c is a nonzero element in U, β is onto. Let $c(f-g) \in (y-b)U[y, D]$. Then $f-g \in c^{-1}(y-b)U[y, D]=(y-a)c^{-1}U[y, D]=(y-a)U[y, D]$. Hence β is one-to-one.

PROPOSITION 2. *If S is a right quotient monoid and* $y-a \in S$, *then* $y-b \in S$ *for all* $b \in U$.

Proof. Since S is a right quotient monoid and $y-a=1(y-a) \in S$, $1 \in S$. If u is a nonzero element in U, $u \cdot u^{-1}=1 \in S$ and so $u \in S$. Since $y-a \in S$ and since $y-b \in U[y, D]$ implies $(y-a)U(y, D] \cap (y-b)S \neq \phi$, there exist $f \in U[y, D]$ and $s \in S$ such that $(y-a)f=$

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$(y-b)s$. Onthe other hand, $c(y-a)=(y-b)c$ for some nonzero c in U. Hence $(y-b)s=c(y-a)c^{-1}s\in S$. This shows that $y-b\in S$.

PROPOSITION 3. *Given $f=\sum_{i=0}^{n} a_i y^i \in U[y, D]$, $a_n=1$, there exist α_i in U, $1\leq i\leq n$, such that $f=(y-\alpha_i)\cdots(y-\alpha_n)$* [2].

Proof. By induction on degree, we shall determine $\alpha, b_i\in U$, $2\leq i\leq n$, such that $f=(y^{n-1}+b_2y^{n-2}+\cdots+b_n)(y-\alpha)$. By expanding this equation, equating coefficients, and eleminating the b_i, an equation of the form $p(x, D(x), \cdots, D^{(n)}(x))=0$ results. Since U is a universal differential field, there exists a desired α in U.

PROPOSISION 4. *If S is a right quotient monoid in $U[y, D]$, then $S=U\setminus\{0\}$ or $S=U[y, D]\setminus\{0\}$.*

Proof. We know $U\setminus\{0\}\subseteq S$(see the proof of Proposition 2). Suppose S contains an element $f\in U[y, D]$ with $\deg(f)=n\geq 1$. Then $f=\sum_{i=0}^{n} a_i y^i=a_n(y-\alpha_1)\cdots(y-\alpha_n)$ for some $\alpha_i\in U$, $1\leq i\leq n$, by Proposition 3. Since $f\in S$ iff $(y-\alpha_1)\cdots(y-\alpha_n)\in S$, $y-\alpha_i\in S$, so $y-a\in S$ for all $a\in U$ by Proposition 2. Hence $S=U[y, D]\setminus\{0\}$ by Proposition 3.

We consider the set $K=RS^{-1}=\{rs^{-1}: r\in R,\ s\in S\}$ where R is an integral domain and S is a right quotient monoid in R. For $s_1, s_2\in S$, $s_1s_2'=s_2s_1'$ for some $s_1', s_2'\in S$ and for $s\in S, r\in R, sr'=rs'$ for some $s'\in S$ and $r'\in R$. Then K is a ring and it is called the right quotient ring of R with respect to S. Since $S=U\setminus\{0\}$ or $S=U[y, D]\setminus\{0\}$ in $U[y, D]$, the right quotient ring of $U[y, D]$ with respect to S is $U[y, D]$ or the right quotient field of $U[y, D]$.

References

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