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DIFFERENTIABILITY OF THE NORM OF $C(S)$

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1. Introduction

In Banach [2] it is shown that the sup norm of $C(S)$ where S is a compact metric space is weakly (or Gateaux) differentiable at a point f in $C(S)$ if and only if the function f peaks at exactly one point of S. In this note we first extend the Banach's result to separating subspaces of $C(S)$, where S is a compact Hausdorff space, which contains constant functions. Then we prove a necessary and sufficient condition in order that the norm of $C(S)$ be strongly (or Fréchet) differentiable at a point f in $C(S)$. Also the smoothness of the norm of certain incomplete normed linear spaces is discussed.

2. Definitions and Preliminary Theorems

Let K denote a convex body in a normed linear space X, i.e., K is a bounded closed convex set with nonempty interior. A point x of K is called a ***smooth point*** of K if there is a unique functional $\varphi\in X^*$ such that $\|\varphi\|=1$ and $\mathrm{Re}\,\varphi(x)=\sup\{\mathrm{Re}\,\varphi(y):y\in K\}$. Let U be the closed unit ball of X, i.e., $U=\{x\in X: \|x\|\leqq 1\}$. Note that a point x of U is a smooth point of U if there is a unique functional φ in X^* such that $\|\varphi\|=1=\varphi(x)$. There are smooth points of U which satisfy a stronger condition. A point x of U is called a ***strong smooth point*** if every sequence of functionals $\{\varphi_n\}$ in X^* satisfying $\|\varphi_n\|\leqq 1$ and $\varphi_n(x)\to 1$ as $n\to\infty$ is convergent in norm to a functional in X^*. Clearly the limit of such a sequence of functionals is the unique functional in the definition of a smooth point of U. Hence, every strongly smooth point of the unit ball is a smooth point of the ball but the converse need not hold. It is well known that the norm of X is weakly (strongly, respectively) differentiable at a point x in X, $\|x\|=1$, if and only if x is a smooth (strongly smooth, respectively) point of U ([7], [4, p. 472]). We say that the norm of a normed linear space is weakly (strongly, respectively) differentiable if every boundary point of the unit ball is smooth (strongly smooth, respectively). A point x of a subset K of a linear space X is called an ***extreme point*** of the subset K if $x=tx_1+(1-t)x_2$, $0<t<1$, x_1, x_2 are points of K, implies $x_1=x=x_2$. We say that a complex valued function f in $C(S)$ peaks at $s\in S$ if $|f(s)|>|f(t)|$ for all $t\in S$ with $t\neq s$, For the real valued functions we simply replace the inequality by $f(s)>f(t)$ [6].

The theorems quoted in this paper are all standard ones and easily accessible, hence we include only a few basic theorems here. In describing the extremal structure of subsets of a Banach space, perhaps, one of the most useful theorems is the Krein Milman theorem. Before stating the theorem we note that every nonempty compact set in a locally convex space has at least an extreme point. The Krein Milman theorem is as follows: *if K is a compact convex subset of a locally convex space and E is the set of extreme points, then K is the closed convex hull of E* ([4], p. 440). Another interesting theorem is that if X is a closed linear subspace of $C(S)$ then every extreme point φ of the closed unit ball B^* of X^* is of the form αs^* where α is a complex

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number with $|\alpha|=1$ and s^* is the evaluation functional of a point $s\in S$ (see [4], p. 441). Here X^* is the space of all continuous linear functionals on a Banach space X. X^* may be equipped with several topologies. One of them is the natural sup norm topology. We will consider the weak* topology defined as follows: Let $\varphi_0\in X^*$, and select a finite number of elements $x_1, x_2, \cdots, x_n\in X$ and a positive number ε. Define

$$U=\{\varphi\in X^*: |\varphi(x_k)-\varphi_0(x_k)|<\varepsilon,\ k=1,2,\cdots\cdots,n\}.$$

This U is a basic neighbourhood of φ_0 and the weak* topology is the collection of unions of these basic neighborhoods. It is straightforward to see that the weak* topology on X^* is locally convex and *the closed unit ball* $B^*=\{\varphi\in X^*: \|\varphi\|\leqq 1\}$ *is compact in this topology*.

3. Differentiability of the norm of subspaces of $C(S)$

THEOREM 1. *Let X be a separating subspace of $C(S)$ with $1\in X$. Then the norm of X is weakly differentiable at a point f in X if and only if f peaks at a point s of S.*

Proof. If the norm of X is weakly differentiable at f with $\|f\|=1$, then f is a smooth point of $B=\{g\in X: \|g\|\leqq 1\}$. Now let r be a member of the set:

$$\{s\in S: |f(s)|=\mathrm{Max}_{t\in S}|f(t)|\}.$$

If r^* represents the evaluation functional at r (see [6]),. then $\varphi=f(r)^{-1}r^*$ is the unique functional in X^* such that $\|\varphi\|=1=\varphi(f)$ and f peaks only at r. Now suppose that f is a function in X with $\|f\|=1=|f(s)|>|f(t)|$ for all $t\in S,\ t\neq s$. Let

$$K=\{\varphi\in X^*: \varphi(f)=1=\|\varphi\|\}.$$

K is clearly nonempty weak* compact convex subset of X^*. And there is an extreme point φ_0 of K. It is easy to see that φ_0 is also an extreme point of the unit ball of X^*. In fact, let $\varphi_0=t\varphi_1+(1-t)\varphi_2$, $0<t<1$, and φ_1, φ_2 are members of the unit ball of X^*. Then $1=\varphi_0(f)=t\varphi_1(f)+(1-t)\varphi_2(f)$. Therefore $\varphi_1(f)=1=\varphi_2(f)$. Hence $\|\varphi_1\|=1=\|\varphi_2\|$ and φ_1, φ_2 belong to K, hence $\varphi_0=\varphi_1=\varphi_2$ and φ_0 is an extreme point of the unit ball of X^*. Now we see that there is a complex number α with $\|\alpha\|=1$ such that $\varphi_0=\alpha t^*$ for some $t\in S$ where t^* denotes the evaluation functional at t. Since $\varphi_0\in K$, $1=\varphi_0(f)=\alpha t^*(f)=\alpha f(t)$ and it follows that $s=t$. Therefore $\varphi_0=f(s)^{-1}s^*$ is the unique extreme point of K and by the Krein Milman theorem $K=\{\varphi_0\}$. Hence f is the smooth point of the unit ball.

COROLLARY. *If a closed separating subspace X of $C(S)$ with $1\in X$ contains a weakly compact fundamental subset K,* i.e., *X is the closure of the set of linear combinations of members of K, then functions peaking at exactly one point form a dense G_δ subset of X.*

Proof. It is known that the norm of a Banach space generated by a weakly compact fundamental subset is weakly differentiable on a dense G_δ subset of the space [1, Theorem 2]. Now the Corollary follows from Theorm 1.

We presented a Banach space which is generated by a weakly compact convex subset in the Corollary. This class of Banach spaces is important because it contains all separable Banach spaces and many properties of separable Banach spaces are shared by the spaces in this class (see [1] and [3]).

The following theorem characterizes the strong differentiability of the norm of $C(S)$.

A similar theorem appears to be incorrectly stated in some literature ([4], p. 472) and we would like to note that by the following theorem the norm of $C(S)$ is nowhere strongly differentiable if the compact set S has no isolated points.

THEOREM 2. *The norm of $C(S)$, S is a compact Hausdorff space, is strongly differentiable at a point f of X if and only if the function f peaks at exactly one point s of S and s is an isolated point of S.*

Proof. Suppose that a function $f \in C(S)$ peaks at an isolated point s_0 of S and that $\varphi_n \in C(S)^*$, $\|\varphi_n\| \leqq 1$ and $\{\varphi_n(f)\}$ converges to $\|f\|$. By the Riesz representation theorem for each n there exists a regular Borel measure μ_n on S such that

$$\varphi_n(f) = \int_S f(s)\, d\mu_n(s).$$

Since s_0 is isolated, $S - \{s_0\}$ is compact, and

$$\|f\| - \sup\{|f(s)| : s \in S - \{s_0\}\} = \delta > 0.$$

Therefore

$$\begin{aligned} \left|\int_S f(s)\, d\mu_n(s)\right| &\leq \left|\int_{S-\{s_0\}} f(s)\, d\mu_n(s)\right| + \left|\int_{\{s_0\}} f(s)\, d\mu_n(s)\right| \\ &\leq (\|f\| - \delta)\|\mu_n\|(S - \{s_0\}) + \|f\|\|\mu_n\|(\{s_0\}) \\ &= \|f\| - \delta\|\mu_n\|(S - \{s_0\}) \end{aligned}$$

where $\|\mu\|(A)$ denotes the total variation of μ over A. Since

$$\varphi_n(f) = \int_S f(s)\, d\mu_n(s) \to \|f\| \quad \text{as } n \to \infty,$$

$\|\mu_n\|(S - \{s_0\}) \to 0$ as $n \to \infty$. Hence $\|\mu_n - \alpha\mu\| \to 0$ as $n \to \infty$ where μ is the unit point mass at s_0 and $\alpha = f(s_0)^{-1}|f(s_0)|$. To prove the converse, suppose that a function f of $C(S)$ peaks at s_0 in S and that there is a sequence $\{s_n\}$ in $S - \{s_0\}$ such that $s_n \to s_0$. For each n choose a continuous function f_n such that $f_n(s_n) = 0$, $f_n(s_0) = 1$ and $\|f_n\| = 1$. Now the sequence $\{s_n^*\}$ of evaluation functionals satisfies $s_n^*(f) \to \|f\|$ and $\|s_n^*\| = 1$ but $\|s_n^* - s_0^*\| \geqq 1$. Hence the norm of $C(S)$ is not strongly differentiable at f.

In the proof of Corollary to Theorem 1 we have mentioned that a Banach space generated by a weakly compact subset (hence a separable Banach space also) has a dense G_δ subset of smooth points on the boundary of the unit ball. Suppose that S is the Cantor's ternary set and let X be the space of all continuous simple functions on S. No function of X peaks at exactly one point of S and none of the boundary points of the unit ball of X is smooth. However, it is known that the above space is not a continuous linear image of a separable Banach space. If a normed linear space is a 1-1 continuous linear image of a separable Banach space, then the unit ball of the space has dense smooth points on its boundary. For example, the subspac of infinitely differentiable functions of $C[0, 1]$ has a dense subset of smooth points (or peaking functions) on the boundary of the unit ball.

THEOREM 3. *Let X be a normed linear space which is a one-one continuous linear image of a separable Banach space, then the smooth points of a convex body K in X form a dense subset of the boundary of K. Hence the norm of X is weakly differentiable on a dense subset of X.*

Proof. Let Y be a separable Banach space, and let $T: Y \to X$ be a 1-1 continuous linear operator from Y onto X. If U is a neighbourhood of a boundary point of K, then $T^{-1}(U)$ is a neighbourhood of a boundary point of the convex body $T^{-1}(K)$ and there is a smooth point y of K in $T^{-1}(U)$ ([5], Theorem 1.7). Clearly $T(y)$ is a smooth point of K and $T(y) \in U$, hence smooth points are dense on the boundary of K.

References

[1] Asplund, E., *Fréchet differentiability of convex functions*, Acta Math. **121** (1968), 31-47,

[2] Banach, S., *Theorie des opérations linéaires*. Monografje Matematyczne, Warsaw, 1932.

[3] Cho, T., *On the Choquet boundary for a nonclosed subspace of* $C(S)$, Pacific J. Math. **35** (1970), 575-580.

[4] Dunford, N. and Schwartz, J., *Linear operators, Part I*, Interscience Publishers, Inc. 1966.

[5] Klee, V., *Some new results on smoothness and rotundity in normed linear spaces*, Math. Ann. **139** (1959), 51-63.

[6] Phelps, R., *Lectures on Choquet's theorem*, Van Nostrand, 1966.

[7] Smulian, V.L., *Sur la dérivabilité de la norm dans l'espace de Banach*, Doklady Akad. Nauk SSSR (N.S.) **27** (1940), 643-648.

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