A Note on Convexity on Linear Vector Space

Suk-Kang Hong*

1. Introduction

Study on convexity has been improved in many statistical fields, such as linear programming, stochastic inventory problems and decision theory. In proof of main theorem in Section 3, M. Loeve already proved this theorem with the r-th absolute moments on page 160 in [1]. Main consideration is given to prove this theorem using convex theorems with the generalized t-th mean when some convex properties hold on a real linear vector space R_N, which satisfies all properties of finite dimensional Hilbert space. Throughout this paper $\underline{x}_j$, $\underline{y}_j$ where $j=1, 2, \ldots\ldots, k, \ldots\ldots, N$, denotes the vectors on R_N, and C_N also denotes a subspace of R_N.

2. Presentation of Definitions and Theorems

Definition 1. Let C_N be a convex set in R_N such that $f(\underline{x}) : C_N \to R_N$. Then a function $f(\underline{x})$ is convex on C_N if and only if for $\underline{x}_1$, $\underline{x}_2 \in C_N$ and $P_1+P_2=1$ where $P_1 \geqq 0$, $P_2 \geqq 0$. There exist $f(P_1\underline{x}_1+P_2\underline{x}_2) \leqq P_1 f(\underline{x}_1)+P_2 f(\underline{x}_2)$

Theorem 1. Let a function $f(\underline{x})$ be convex on R_N and for

$$\sum_{j=1}^{k} p_j=1 \quad \text{where } p_j \geqq 0 \quad \text{for } j=1, 2, \ldots\ldots K.$$

If $\underline{x}_1$, $\underline{x}_2, \ldots\ldots \underline{x}_k \in R_N$ exist

then $$f\left(\sum_{j=1}^{k} P_j \underline{x}_j\right) \leqq \sum_{j=1}^{k} P_j f(\underline{x}_j)$$

〈Proof〉 Proof is complete by mathematical induction if we prove the Definition 1 for $j=1$.

Theorem 2. The intersection of any numbers of convex sets is also convex.

* The author is Lecturer of Statistics, Department of Statistics, Dongguk University, Seoul, Korea.

〈Proof〉 Let P_1, P_2 be such that $P_1+P_2=1$

where $P_1 \geqq 0$, $P_2 \geqq 0$.

Let A and B be convex sets such that $\underline{x}_1, \underline{x}_2 \in A \cap B$.

Then $\underline{x}_1 \in A$, $\underline{x}_2 \in A$ and $\underline{x}_1 \in B$, $\underline{x}_2 \in B$.

Therefore it holds that $P_1\underline{x}_1+P_2\underline{x}_2 \in A$

from convexity of A and $P_1\underline{x}_1+P_2\underline{x}_2 \in B$

from convexity of B.

So $P_1\underline{x}_1+P_2\underline{x}_2 \in A \cap B$ Q. E. D.

Corollary 1. The intersection of linear vector space C_N in R_N is a linear vector space which is itself convex.

Corollary 2. Every linear vector space is a convex set.

〈Proof〉 Proofs for corollary 1 and corollary 2 are detailed in [3].

Theorem 3. If a function $f_j(\underline{x})$ is convex on C_N where a vector $\theta_j \geqq 0$ for $j=1, 2, \ldots\ldots K$.

then $\sum_{j=1}^{k} \theta_j f_j(\underline{x})$ is convex on C_N for all j.

〈Proof〉 Since for $\theta_j \geqq 0$, $f_j(\underline{x})$ is convex on C_N for all j,

there exist $\underline{x}_1$, $\underline{x}_2 \in C_N$ such that $f_j(P_1\underline{x}_1+P_2\underline{x}_2) \leqq P_1 f_j(\underline{x}_1)+P_2 f_j(\underline{x}_2)$ by Definition 1.

Therefore $\theta_j f_j(P_1\underline{x}_1+P_2\underline{x}_2) \leqq \theta_j[P_1 f_j(\underline{x}_1)+P_2 f_j(\underline{x}_2)]$

$$=P_1\theta_j f_j(\underline{x}_1)+P_2\theta_j f_j(\underline{x}_2),$$

that is,

$$[\theta_j f_j(P_1 x_1+P_2 x_2)] \leqq P_1[\theta_j f_j(x_1)]+P_2[\theta_j f_j(x_2)].$$

So $\theta_j f_j(x)$ is convex on C_N for all j.

We can extend this property to $j=k$

by mathematical induction

$$\sum_{j=1}^{k}\theta_j f_j(P_1\underline{x}_1+P_2\underline{x}_2) \leqq P_1[\sum_{j=1}^{k}\theta_j f_j(\underline{x}_1)]+P_2[\sum_{j=1}^{k}\theta_j f_j(\underline{x}_2)].$$

Thus $\sum_{j=1}^{k}\theta_j f_j(\underline{x})$ is convex on C_N. Q. E. D.

Theorem 4. Suppose we have a set $\{f_j(\underline{x})\}$ of convex functions on C_N.

Let $g(\underline{x})=\max_j\{f_j(\underline{x})\}$,

then $g(x)$ is a convex function on C_N.

〈Proof〉 Let's take $\underline{x}_1$, $\underline{x}_2 \in C_N$. Then $f_j(P_1\underline{x}_1+P_2\underline{x}_2) \leqq P_1 f_j(\underline{x}_1)+P_2 f_j(\underline{x}_2)$ for each j.

Hence $\max_j f_j(P_1\underline{x}_1+P_2\underline{x}_2) \leqq \max_j \{P_1 f_j(\underline{x}_1)+P_2 f_j(\underline{x}_2) \leqq \max_j \{P_1 f_j(\underline{x}_1)\}$

$+\max_j \{P_2 f_j(\underline{x}_2)\} = P_1 \max_j \{f_j(\underline{x}_1)\} + P_2 \max_j \{f_j(\underline{x}_2)\}$.

Thus $g(P_1 x_1+P_2 x_2 \leqq P_1 g(x_1)+P_2 g(x_2)$. Q. E. D.

Theorem 5. If a function $f(x)$ is convex on C_N and $a \leqq f(x) \leqq b$ on C_N, If $h(\underline{y})$ is convex on $[a, b]$ and $h(\underline{y})$ is monotonically increasing, then $g(\underline{x}) = h\{f(\underline{x})\}$ is convex on C_N.

〈Proof〉 Let's take $\underline{x}_1$, $\underline{x}_2 \in C_N$. Then $a \leqq f(\underline{x}_1) \leqq b$ and $a \leqq f(\underline{x}_2) \leqq b$.

Hence $a \leqq P_1 f(\underline{x}_1)+P_2 f(\underline{x}_2) \leqq b$.

Since $h(\underline{y})$ is monotonically increasing

and since $f(P_1\underline{x}_1+P_2\underline{x}_2) \leqq P_1 f(\underline{x}_1)+P_2 f(\underline{x}_2)$

we have $g[P_1\underline{x}_1+P_2\underline{x}_2] \leqq h[P_1 f(\underline{x}_1)+P_2 f(\underline{x}_2)]$.

Since $h(\underline{y})$ is convex on $[a, b]$

we have $h[P_1 f(\underline{x}_1)+P_2 f(\underline{x}_2)] \leqq P_1 g(\underline{x}_1)+P_2 g(\underline{x}_2)$.

Therefore $g(P_1\underline{x}_1+P_2\underline{x}_2) \leqq P_1 g(\underline{x}_1)+P_2 g(\underline{x}_2)$. Q. E. D

Definition 2. Let $\underline{x}$, $\underline{P}$ be elements of R_N.

Then for $\underline{P} \geqq 0$ $\underline{x} \geqq 0$ and $\sum_{j=1}^{k} P_j = 1$,

$$M_t(\underline{x}, \underline{P}) = \left\{\sum_{j=1}^{k} P_{jj} x^t\right\}^{\frac{1}{t}} \quad \text{for} \quad -\infty < t < \infty$$

may be called Generalized t-th Mean.

Examples. (1) If we put $P_j = \frac{1}{k}$, $t=1$,

then $M_t(\underline{x}, \underline{P}) = \left\{\frac{1}{k}\sum_{j=1}^{k} x_j\right\}$: arithmetic mean.

(2) If we put $P_j = \frac{1}{k}$, $t=2$,

$M_2 = M_t(\underline{x}, \underline{P}) = \left\{\frac{1}{k}\sum_{i=1}^{k} x_j^2\right\}^{\frac{1}{2}}$: root mean square.

(3) If we put $P_j = \frac{1}{k}$, $t=-1$,

$M_t(\underline{x}, \underline{P}) = \dfrac{1}{\frac{1}{k}\sum_{i=1}^{k}\frac{1}{x_j}}$: harmonic mean.

(4) If we put $t=1$,

$M_1 = M_t(\underline{x}, \underline{P}) = \{\sum_{j=1}^{k} P_j x_j\}$: generalized arithmetic mean.

(5) If we put $t=-1$,

$$M_{-1}=M_t(\underline{x},\underline{P})=\left\{\sum_{j=1}^{k}\frac{P_j}{x_j}\right\}^{-1}:\ \text{generalized harmonic mean.}$$

(6) $M_0=\lim_{t\to 0} M_t(\underline{x},\underline{P})=\lim_{t\to 0} e^{\frac{1}{t}\log\sum_{j=1}^{k}p_jx_j^t}$

$=e^{\sum_{j=1}^{k}p_j\log x_j}$ (by L'Hospital's Rule)

$=\prod_{j=1}^{k}x_j^{p_j}$: generalized geometric mean.

Observation: Let $x_k=\max_j\{x_j\}$.

Then $(P_kx_k)^{\frac{1}{t}}\leqq\{\sum_{j=1}^{k}P_jx_j^t\}\leqq x_k$,

that is, $P_k^{\frac{1}{t}}x_k\leqq M_t(\underline{x},\underline{P})\leqq x_k$ as $t\to\infty$.

We have $M_\infty(\underline{x},\underline{P})=\lim_{t\to\infty}M_t(\underline{x},\underline{P})=x_k$, and

also, we have $M_{-t}(\underline{x},\underline{P})=\dfrac{1}{M_t(\underline{x}^{-1},\underline{P})}$

$$=\frac{1}{\max_j\{x_j^{-1}\}}=\min_j\{x_j\}\ \text{as}\ t\to\infty.$$

3. Main Theorem

Theorem 6. A generalized t-th mean $M_t(\underline{x},\underline{P})$ is monotonically increasing on $-\infty<t<\infty$ and also $M_t(\underline{x},\underline{P})$ is strictly increasing unless x_j's are equal.

This theorem when provided will imply

$$\min_j\{x_j\}\leqq M_{-1}\leqq M_0\leqq M_1\leqq M_2\leqq\max_j\{x_j\},$$

where M_{-1}, M_0, M_1, and M_2 are represented in the above examples. And this is also similar with M. Lóeveś Theorem in [1].

Example 2. Let x be a nonzero discrete random variable.

Then $|x|$ takes on k positive values. Hence

$$\min\{|x|\}\leqq\frac{1}{E\{|x|^{-1}\}}\leqq e^{E\{\log|x|\}}\leqq E\{|x|\}\leqq E^{\frac{1}{2}}\{|x|^2\}\leqq\max\{|x|\}.$$

Hence $E\{|x|\}E\{|x|^{-1}\}\geqq 1$.

⟨Proof of Main Theorem⟩

We need the following five lemmas.

Lemma 1. If there exist $P_1, P_2\geqq 0$ where $P_1+P_2=1$ and $x_2\geqq x_1$, then $P_1x_1+P_2x_2\geqq x_1$.

〈Proof〉 We have $P_2(x_2-x_1)\geqq 0$

but $P_2(x_2-x_1)=P_1x_1+P_2x_2-(P_1+P_2)x_1=P_1x_1+P_2x_2-x_1$.

Hence $P_1x_1+P_2x_2\geqq x_1$. Q. E. D.

Lemma 2. If a function $f(b)$ is non-decreasing and integrable on $[0, a]$

then $\frac{1}{t}\int_0^t f(u)\,du$ is non-decreasing.

That is $\frac{1}{t}\int_0^t dF(u)$ is non-decreasing.

〈Proof〉 Let $B=\frac{1}{t_2}\int_0^{t_2} f(u)\,du$ where $0<t_1<t_2<a$,

then $B=\frac{t_1}{t_1+(t_2-t_1)}\left\{\frac{1}{t_1}\int_0^{t_1} f(u)\,du\right\}+\frac{t_2-t_1}{t_1+(t_2-t_1)}\left\{\frac{1}{t_2-t_1}\int_{t_1}^{t_2} f(u)\,du\right\}$.

Let $P_1=\frac{t_1}{t_1+(t_2-t_1)}$ and $P_2=\frac{t_2-t_1}{t_1+(t_2-t_1)}$.

Then $P_1+P_2=1$ we can get.

Since $f(t)$ is increasing and $t_2>t_1>0$.

we always have

$$\frac{1}{t_2-t_1}\int_{t_1}^{t_2} f(u)\,du\geqq\frac{1}{t_1}\int_0^{t_1} f(u)\,du \text{ from [6].}$$

Again put $x_1=\frac{1}{t_1}\int_0^{t_1} f(u)\,du$ and $x_2=\frac{1}{t_2-t_1}\int_{t_1}^{t_2} f(u)\,du$.

By Lemma 1, $B=P_1x_1+P_2x_2\geqq x_1$.

Therefore Lemma 2 was proved. Q. E. D.

Lemma 3. If $f_j(\underline{x})$ is increasing for $j=1, 2\cdots\cdots\cdots\cdots, k$.

then $\sum_{j=1}^{k} f_j(\underline{x})$ is increasing.

Proof is trivial.

Definition 3. A function $f(\underline{x})$ is log convex if and only if $f(\underline{x})=e^{u(x)}$ where $u(\underline{x})$ is convex.

Lemma 4. A function $f(\underline{x})$ is log convex if and only if $f(p\underline{x}+q\underline{y})\leqq f^p(\underline{x})f^q(\underline{y})$ where $p, q\geqq 0$ and $p+q=1$.

〈Proof〉 $f(p\underline{x}+q\underline{y})=e^{u\{p(x)+q(y)\}}\leqq e^{pu(x)+qu(y)}=f^p(\underline{x})f^q(\underline{y})$.

Lemma 5. If $f(\underline{x})$ and $q(\underline{x})$ are log convex, then $f(\underline{x})+q(\underline{x})$ is also log convex.

〈Proof〉 From the Hölder Inequality, we have

$\boldsymbol{a}, \boldsymbol{b}\geqq 0$ and $\boldsymbol{a}, \boldsymbol{b}\leqq N^{\frac{1}{p}}(a), N^{\frac{1}{q}}(b)$

for $p, q\geqq 0$ and $p+q=1$

where $\underline{a}, \underline{b} \in R_N$,

$N_{\frac{1}{p}}(a)=\left\{\sum_{j=1}^{N}|q_j|^p\right\}^{\frac{1}{p}}$ and $N_{\frac{1}{q}}(b)=\left\{\sum_{j=1}^{N}|b_j|^q\right\}^{\frac{1}{q}}$.

In R_2,

we may have

$a_1b_1+a_2b_2 \leqq [a_1^{\frac{1}{p}}+a_2^{\frac{1}{p}}]^p[b_1^{\frac{1}{q}}+b_2^{\frac{1}{q}}]^q$

for $\underline{a}, \underline{b} \geqq 0$.

Let $a_1=A^p(\underline{x})$ $a_2=B^p(\underline{x})$

$b_1=A^q(\underline{y})$ $b_2=B^q(\underline{y})$.

Then the inequality implies

$$A^p(\underline{x})A^q(\underline{y})+B^p(\underline{x})B^q(\underline{y}) \leqq [A(\underline{x})+B(\underline{x})]^p[A(\underline{y})+B(\underline{y})]^q.$$

Suppose $A(\underline{x})$ and $B(\underline{x})$ are log convex on C_N

and let's put $H(\underline{x})=A(\underline{x})+B(\underline{x})$,

$H(p\underline{x}+q\underline{y})=A(p\underline{x}+q\underline{y})+B(p\underline{x}+q\underline{y})$

$\leqq A^p(\underline{x})A^q(\underline{y})+B^p(\underline{x})B^q(\underline{y})$

$\leqq [A(\underline{x})+B(\underline{x})]^p[A(\underline{y})+B(\underline{y})]^q$

$=H^p(\underline{x})H^q(\underline{y})$.

Hence $H(p\underline{x}+q\underline{y}) \leqq H^p(\underline{x})H^q(\underline{y})$.

Thus $H(\underline{x})=A(\underline{x})+B(\underline{x})$ is log convex by Lemma 4.

Therefore if we put $A(\underline{x})=f(\underline{x})$, $B(\underline{x})=q(\underline{x})$,

then the proof is complete. Q. E. D.

Corollary 3. If $f_j(\underline{x})$ is log convex on C_N in R_N for $j=1, 2 ...... K$ and $\theta_j>0$,

then $\sum_{j=1}^{k}\theta_j f_j(\underline{x})$ is log convex on C_N.

With help of these five lemmas and corollary 3,

since x_j^t is log convex in $t \in [-\infty, +\infty]$

$\sum_{j=1}^{k}P_j x_j^t$ is log convex in t, and

also by lemma 2, since $\left\{\frac{d}{dt}\log\sum_{j=1}^{k}P_j x_j^t\right\}$ is increasing

$$M_t(\underline{xP})=\left\{\sum_{j=1}^{k}P_j x_j^t\right\}^{\frac{1}{t}}=\exp^{\frac{1}{t}\log\left[\sum_{j=1}^{k}p_j x_j t\right]}$$

by L' Hospital's Rule

$$=\exp\left\{\frac{1}{t}\int_0^t\frac{d(\log\sum_{j=1}^{k}P_j x_j^y)}{dy}dt\right\}.$$

But $\frac{d}{dy}\left\{\log \sum_{j=1}^{k} P_j x_j^{y}\right\}$ is increasing.

Hence $M_t(\underline{x}, \underline{P})$ is increasing in $t \in [-\infty, +\infty]$. Q. E. D.

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