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NOTE ON $(f, U_{(k)}, u_{(k)}, \alpha_{(k)})$-STRUCTURES

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§0. Introduction

Submanifolds of codimension 2 in an almost contact manifold and hypersurfaces of a manifold with (f, g, u, v, λ)-structure admit an $(f, U_{(k)}, u_{(k)}, \alpha_{(k)})$-structure [1], [3], [6].

Ki, Pak and Suh have studied a manifold with $(f, U_{(k)}, u_{(k)}, \alpha_{(k)})$-structure and hypersurfaces of an even-dimensional sphere in terms of this structure [1].

The main purpose of the present paper is to study the integrability conditions of this structure.

In §1, we recall the definitions of $(f, U_{(k)}, u_{(k)}, \alpha_{(k)})$ and $(f, g, u_{(k)}, \alpha_{(k)})$-structure.

In §2, we study the integrability conditions of $(f, U_{(k)}, u_{(k)}, \alpha_{(k)})$-structure.

In the last §3, we investigate the necessary and sufficient condition for product Riemannian manifold $M\times R^3$ to be Kählerian.

§1. Preliminaries

Let M be an m-dimensional differentiable manifold of class C^∞. If there exist on M a (1,1) tensor field f, vector fields U, V and W, 1-forms u, v and w and differentiable functions α, β and γ satisfying the following conditions (1.1)~(1.6). Then we say that M has an $(f, U_{(k)}, u_{(k)}, \alpha_{(k)})$-structure# [1] ;

(1.1) $$f^2=-I+u\otimes U+v\otimes V+w\otimes W,$$

I being the unit tensor field of type (1,1),

(1.2) $$fU=-\gamma V+\beta W,\quad fV=\gamma U+\alpha W,\quad fW=-\beta U-\alpha V,$$

(1.3) $$u\circ f=\gamma v-\beta w,\quad v\circ f=-\gamma u-\alpha w,\quad w\circ f=\beta u+\alpha v,$$

where 1-form $u_{(k)}\circ f$ is defined by $(u_{(k)}\circ f)(X)=u_{(k)}(fX)$ $(k=1,2,3)$ for any vector field X, and

#We denote by $U_{(1)}=U$, $U_{(2)}=V$, $U_{(3)}=W$, $u_{(1)}=u$, $u_{(2)}=v$, $u_{(3)}=w$, $\alpha_{(1)}=\alpha$, $\alpha_{(2)}=\beta$ and $\alpha_{(3)}=\gamma$.

(1.4) $u(U)=1-\beta^2-\gamma^2,\ u(V)=-\alpha\beta,\ u(W)=-\alpha\gamma,$

(1.5) $v(U)=-\alpha\beta,\ v(V)=1-\alpha^2-\gamma^2,\ v(W)=\beta\gamma,$

(1.6) $w(U)=-\alpha\gamma,\ w(V)=\beta\gamma,\ w(W)=1-\alpha^2-\beta^2.$

It is well known that any submanifold of codimension 2 immersed in an almost contact manifold and any hypersurface immersed in a manifold with (f, U, V, u, v, λ)-structure admit an $(f, U_{(k)}, u_{(k)}, \alpha_{(k)})$-structure (cf. [1], [3], [6] etc.). It is verified that the dimension of a manifold with $(f, U_{(k)}, u_{(k)}, \alpha_{(k)})$-structure is odd [1]. If a manifold with $(f, U_{(k)}, u_{(k)}, \alpha_{(k)})$-structure has a positive definite Riemannian metric g satisfying the conditions;

(1.7) $g(U, X)=u(X),\ g(V, X)=v(X),\ g(W, X)=w(X),$

(1.8) $g(fX, fY)=g(X, Y)-u(X)u(Y)-v(X)v(Y)-w(X)w(Y)$

for any vector fields X and Y, then we say that M has an $(f, g, u_{(k)}, \alpha_{(k)})$-structure [1]. Any submanifold of codimension 2 immersed in an almost contact metric manifold and any hypersurface immersed in a manifold with (f, g, u, v, λ)-structure admit an $(f, g, u_{(k)}, \alpha_{(k)})$-structure (cf. [1], [3], [6] etc.).

§2. An almost complex structure on $M\times R^3$

Suppose that an m-dimensional manifold M has an $(f, U_{(k)}, u_{(k)}, \alpha_{(k)})$-structure. Now, we consider the product manifold $M\times R^3$, R^3 being a three dimensional Euclidean space. We define on $M\times R^3$ a tensor field F of type (1,1) with local components $F_B{}^A$ given by

$$(2.1)\qquad (F_B{}^A)=\begin{pmatrix} f_c^a & U^a & V^a & W^a \\ -u_c & 0 & -\gamma & \beta \\ -v_c & \gamma & 0 & \alpha \\ -w_c & -\beta & -\alpha & 0 \end{pmatrix}$$

in $\{N\times R^3, X^A\}$, $\{N, X^a\}$ being coordinate neighborhood of M and $X^{\bar 1}, X^{\bar 2}, X^{\bar 3}$ being cartesian coordinate in R^3, where here and in the sequel the indices $A, B, C, \cdots$ run over the range $\{1, 2, 3, \cdots, m, \bar 1, \bar 2, \bar 3\}$ and $a, b, c, \cdots$ run over the range $\{1, 2, 3, \cdots, m\}$ and $f_c{}^a$, U^a, V^a, W^a, u_c, v_c and w_c are respectively components of f, U, V, W, u, v, w in $\{N, X^a\}$. Then, taking account of (1.1)~(1.6), we can see that $F^2=-I$ holds on $M\times R^3$. Thus we have

PROPOSITION 2.1. *If there is given an $(f,\ U_{(k)},\ u_{(k)},\ \alpha_{(k)})$-structure on M, then the tensor field F defined by* (2.1) *is an almost complex structure on $M\times R^3$.*

The Nijenhuis tensor $[F,F]$ of F has local components

$$[F,F]_{CB}{}^{A}=F_C{}^{E}\partial_E F_B{}^{A}-F_B{}^{E}\partial_E F_C{}^{A}-(\partial_C F_B{}^{E}-\partial_B F_C{}^{E})F_E{}^{A} \tag{2.2}$$

on $M\times R^3$. Thus, using (2.1), we can write down $[F,F]_{CB}{}^{A}$ as follows:

$$(2.3)_1\quad [F,F]_{cb}{}^{a}=f_c{}^{e}\partial_e f_b{}^{a}-f_b{}^{e}\partial_e f_c{}^{a}-(\partial_c f_b{}^{e}-\partial_b f_c{}^{e})f_e{}^{a}+(\partial_c u_b-\partial_b u_c)U^a+(\partial_c v_b-\partial_b v_c)V^a+(\partial_c w_b-\partial_b w_c)W^a,$$

$$(2.3)_2\quad [F,F]_{cb}{}^{\bar{1}}=-f_c{}^{e}\partial_e u_b+f_b{}^{e}\partial_e u_c+(\partial_c f_b{}^{e}-\partial_b f_c{}^{e})u_e-\gamma(\partial_c v_b-\partial_b v_c)+\beta(\partial_c w_b-\partial_b w_c),$$

$$(2.3)_3\quad [F,F]_{cb}{}^{\bar{2}}=-f_c{}^{e}\partial_e v_b+f_b{}^{e}\partial_e v_c+(\partial_c f_b{}^{e}-\partial_b f_c{}^{e})v_e+\gamma(\partial_c u_b-\partial_b u_c)+\alpha(\partial_c w_b-\partial_b w_c),$$

$$(2.3)_4\quad [F,F]_{cb}{}^{\bar{3}}=-f_c{}^{e}\partial_e w_b+f_b{}^{e}\partial_e w_c+(\partial_c f_b{}^{e}-\partial_b f_c{}^{e})w_e-\beta(\partial_c u_b-\partial_b u_c)-\alpha(\partial_c v_b-\partial_b v_c),$$

$$(2.3)_5\quad [F,F]_{c\bar{1}}{}^{a}=f_c{}^{e}\partial_e U^a-U^e\partial_e f_c{}^{a}-(\partial_c U^e)f_e{}^{a}-(\partial_c\gamma)V^a+(\partial_c\beta)W^a,$$

$$(2.3)_6\quad [F,F]_{c\bar{2}}{}^{a}=f_c{}^{e}\partial_e V^a-V^e\partial_e f_c{}^{a}-(\partial_c V^e)f_e{}^{a}+(\partial_c\gamma)U^a+(\partial_c\alpha)W^a,$$

$$(2.3)_7\quad [F,F]_{c\bar{3}}{}^{a}=f_c{}^{e}\partial_e W^a-W^e\partial_e f_c{}^{a}-(\partial_c W^e)f_e{}^{a}-(\partial_c\beta)U^a-(\partial_c\alpha)V^a,$$

$$(2.3)_8\quad [F,F]_{\bar{1}\bar{2}}{}^{a}=U^e\partial_e V^a-V^e\partial_e U^a,$$

$$(2.3)_9\quad [F,F]_{\bar{1}\bar{3}}{}^{a}=U^e\partial_e W^a-W^e\partial_e U^a,$$

$$(2.3)_{10}\quad [F,F]_{\bar{2}\bar{3}}{}^{a}=V^e\partial_e W^a-W^e\partial_e V^a,$$

$$(2.3)_{11}\quad [F,F]_{c\bar{1}}{}^{\bar{1}}=U^e\partial_e u_c+(\partial_c U^e)u_e+(\partial_c\gamma)\gamma+(\partial_c\beta)\beta,$$

$$(2.3)_{12}\quad [F,F]_{c\bar{1}}{}^{\bar{2}}=f_c{}^{e}\partial_e\gamma+U^e\partial_e v_c+(\partial_c U^e)v_e+(\partial_c\beta)\alpha,$$

$$(2.3)_{13}\quad [F,F]_{c\bar{1}}{}^{\bar{3}}=-f_c{}^{e}\partial_e\beta+U^e\partial_e w_c+(\partial_c U^e)w_e+(\partial_c\gamma)\alpha,$$

$$(2.3)_{14}\quad [F,F]_{c\bar{2}}{}^{\bar{1}}=-f_c{}^{e}\partial_e\gamma+V^e\partial_e u_c+(\partial_c V^e)u_e+(\partial_c\alpha)\beta,$$

$$(2.3)_{15}\quad [F,F]_{c\bar{2}}{}^{\bar{2}}=V^e(\partial_e v_c)+(\partial_c V^e)v_e+(\partial_c\gamma)\gamma+(\partial_c\alpha)\alpha,$$

$$(2.3)_{16}\quad [F,F]_{c\bar{2}}{}^{\bar{3}}=-f_c{}^{e}\partial_e\alpha+V^e\partial_e w_c+(\partial_c V^e)w_e-(\partial_c\gamma)\beta,$$

$$(2.3)_{17}\quad [F,F]_{c\bar{3}}{}^{\bar{1}}=f_c{}^{e}\partial_e\beta+W^e\partial_e u_c+(\partial_c W^e)u_e+(\partial_c\alpha)\gamma,$$

$(2.3)_{18}$ $\qquad [F,F]_{c\bar{3}}{}^{\bar{2}} = f_c{}^e \partial_e \alpha + W^e \partial_e v_c + (\partial_c W^e) v_e - (\partial_c \beta)\gamma,$

$(2.3)_{19}$ $\qquad [F,F]_{c\bar{3}}{}^{\bar{3}} = W^e \partial_e w_c + (\partial_c W^e) w_e + (\partial_c \beta)\beta + (\partial_c \alpha)\alpha,$

$(2.3)_{20}$ $\qquad [F,F]_{\bar{1}\bar{2}}{}^{\bar{1}} = -U^e \partial_e \gamma,$

$(2.3)_{21}$ $\qquad [F,F]_{\bar{1}\bar{2}}{}^{\bar{2}} = -V^e \partial_e \gamma,$

$(2.3)_{22}$ $\qquad [F,F]_{\bar{1}\bar{2}}{}^{\bar{3}} = -U^e \partial_e \alpha + V^e \partial_e \beta,$

$(2.3)_{23}$ $\qquad [F,F]_{\bar{1}\bar{3}}{}^{\bar{1}} = U^e \partial_e \beta,$

$(2.3)_{24}$ $\qquad [F,F]_{\bar{1}\bar{3}}{}^{\bar{2}} = U^e \partial_e \alpha - W^e \partial_e \gamma,$

$(2.3)_{25}$ $\qquad [F,F]_{\bar{1}\bar{3}}{}^{\bar{3}} = W^e \partial_e \beta,$

$(2.3)_{26}$ $\qquad [F,F]_{\bar{2}\bar{3}}{}^{\bar{1}} = V^e \partial_e \beta + W^e \partial_e \gamma,$

$(2.3)_{27}$ $\qquad [F,F]_{\bar{2}\bar{3}}{}^{\bar{2}} = V^e \partial_e \alpha,$

$(2.3)_{28}$ $\qquad [F,F]_{\bar{2}\bar{3}}{}^{\bar{3}} = W^e \partial_e \alpha.$

We can easily verify that there exist on M a tensor field T of type (1,2) with components $[F,F]_{cb}{}^a$, three tensor fields P_1, P_2 and P_3 of type (0,2) with components $[F,F]_{cb}{}^{\bar{1}}$, $[F,F]_{cb}{}^{\bar{2}}$ and $[F,F]_{cb}{}^{\bar{3}}$ respectively, three tensor fields Q_1, Q_2 and Q_3 of type (1,1) with components $[F,F]_{\bar{1}b}{}^a$, $[F,F]_{\bar{2}b}{}^a$ and $[F,F]_{\bar{3}b}{}^a$, three vector fields S_1, S_2 and S_3 with components $[F,F]_{\bar{1}\bar{2}}{}^a$, $[F,F]_{\bar{1}\bar{3}}{}^a$ and $[F,F]_{\bar{2}\bar{3}}{}^a$, nine 1-forms w_1, w_2, w_3, w_4, w_5, w_6, w_7, w_8 and w_9 with components $[F,F]_{c\bar{1}}{}^{\bar{1}}$, $[F,F]_{c\bar{1}}{}^{\bar{2}}$, $[F,F]_{c\bar{1}}{}^{\bar{3}}$, $[F,F]_{c\bar{2}}{}^{\bar{1}}$, $[F,F]_{c\bar{2}}{}^{\bar{2}}$, $[F,F]_{c\bar{2}}{}^{\bar{3}}$, $[F,F]_{c\bar{3}}{}^{\bar{1}}$, $[F,F]_{c\bar{3}}{}^{\bar{2}}$ and $[F,F]_{c\bar{3}}{}^{\bar{3}}$ and nine functions $K_1 = [F,F]_{\bar{1}\bar{2}}{}^{\bar{1}}$, $K_2 = [F,F]_{\bar{1}\bar{2}}{}^{\bar{2}}$, $K_3 = [F,F]_{\bar{1}\bar{2}}{}^{\bar{3}}$, $K_4 = [F,F]_{\bar{1}\bar{3}}{}^{\bar{1}}$, $K_5 = [F,F]_{\bar{1}\bar{3}}{}^{\bar{2}}$, $K_6 = [F,F]_{\bar{1}\bar{3}}{}^{\bar{3}}$, $K_7 = [F,F]_{\bar{2}\bar{3}}{}^{\bar{1}}$, $K_8 = [F,F]_{\bar{2}\bar{3}}{}^{\bar{2}}$ and $K_9 = [F,F]_{\bar{2}\bar{3}}{}^{\bar{3}}$.

We know that the Nijenhuis tensor $[F,F]$ of an almost complex structure F satisfies the conditions

(2.4) $\qquad [F,F]_{CE}{}^A F_B{}^E + [F,F]_{CB}{}^E F_E{}^A = 0$ (cf. [7]).

Substituting (2.1) into (2.4), we have

$(2.5)_1$ $\qquad [F,F]_{ce}{}^a f_b{}^e + [F,F]_{cb}{}^e f_e{}^a - [F,F]_{c\bar{1}}{}^a u_b + [F,F]_{cb}{}^{\bar{1}} U^a - [F,F]_{c\bar{2}}{}^a v_b$

$$+[F,F]_{cb}{}^{\bar{2}}V^{a}-[F,F]_{c\bar{3}}{}^{a}w_{b}+[F,F]_{cb}{}^{\bar{3}}W^{a}=0,$$

$(2.5)_2$

$$\begin{aligned}&[F,F]_{ce}{}^{\bar{1}}f_{b}{}^{e}-[F,F]_{cb}{}^{e}u_{e}-[F,F]_{c\bar{1}}{}^{\bar{1}}u_{b}-[F,F]_{c\bar{2}}{}^{\bar{1}}v_{b}-\gamma[F,F]_{cb}{}^{\bar{2}}\\&\quad-[F,F]_{c\bar{3}}{}^{\bar{1}}w_{b}+\beta[F,F]_{cb}{}^{\bar{3}}=0,\end{aligned}$$

$(2.5)_3$

$$\begin{aligned}&[F,F]_{ce}{}^{\bar{2}}f_{b}{}^{e}-[F,F]_{cb}{}^{e}v_{e}-[F,F]_{c\bar{1}}{}^{\bar{2}}u_{b}+\gamma[F,F]_{cb}{}^{\bar{1}}-[F,F]_{c\bar{2}}{}^{\bar{2}}v_{b}\\&\quad-[F,F]_{c\bar{3}}{}^{\bar{2}}w_{b}+\alpha[F,F]_{cb}{}^{\bar{3}}=0,\end{aligned}$$

$(2.5)_4$

$$\begin{aligned}&[F,F]_{ce}{}^{\bar{3}}f_{b}{}^{e}-[F,F]_{cb}{}^{e}w_{e}-[F,F]_{c\bar{1}}{}^{\bar{3}}u_{b}-\beta[F,F]_{cb}{}^{\bar{1}}-[F,F]_{c\bar{2}}{}^{\bar{3}}v_{b}\\&\quad-\alpha[F,F]_{cb}{}^{\bar{2}}-[F,F]_{c\bar{3}}{}^{\bar{3}}w_{b}=0,\end{aligned}$$

$(2.5)_5$

$$\begin{aligned}&[F,F]_{ce}{}^{a}U^{e}+[F,F]_{c\bar{1}}{}^{e}f_{e}{}^{a}+[F,F]_{c\bar{1}}{}^{\bar{1}}U^{a}+\gamma[F,F]_{c\bar{2}}{}^{a}+[F,F]_{c\bar{1}}{}^{\bar{2}}V^{a}\\&\quad-\beta[F,F]_{c\bar{3}}{}^{a}+[F,F]_{c\bar{1}}{}^{\bar{3}}W^{a}=0,\end{aligned}$$

$(2.5)_6$

$$\begin{aligned}&[F,F]_{ce}{}^{a}V^{e}+[F,F]_{c\bar{2}}{}^{e}f_{e}{}^{a}-\gamma[F,F]_{c\bar{1}}{}^{a}+[F,F]_{c\bar{2}}{}^{\bar{1}}U^{a}+[F,F]_{c\bar{2}}{}^{\bar{2}}V^{a}\\&\quad-\alpha[F,F]_{c\bar{3}}{}^{a}+[F,F]_{c\bar{2}}{}^{\bar{3}}W^{a}=0,\end{aligned}$$

$(2.5)_7$

$$\begin{aligned}&[F,F]_{ce}{}^{a}W^{e}+[F,F]_{c\bar{3}}{}^{e}f_{e}{}^{a}+\beta[F,F]_{c\bar{1}}{}^{a}+[F,F]_{c\bar{3}}{}^{\bar{1}}U^{a}+\alpha[F,F]_{c\bar{2}}{}^{a}\\&\quad+[F,F]_{c\bar{3}}{}^{\bar{2}}V^{a}+[F,F]_{c\bar{3}}{}^{\bar{3}}W^{a}=0,\end{aligned}$$

$(2.5)_8$

$$\begin{aligned}&[F,F]_{\bar{1}e}{}^{a}f_{b}{}^{e}+[F,F]_{\bar{1}b}{}^{e}f_{e}{}^{a}+[F,F]_{\bar{1}b}{}^{\bar{1}}U^{a}-[F,F]_{\bar{1}\bar{2}}{}^{a}v_{b}\\&\quad+[F,F]_{\bar{1}\bar{2}}{}^{b}V_{a}-[F,F]_{\bar{1}\bar{3}}{}^{a}w_{b}+[F,F]_{\bar{1}b}{}^{\bar{3}}W^{a}=0,\end{aligned}$$

$(2.5)_9$

$$\begin{aligned}&[F,F]_{\bar{2}e}{}^{a}f_{b}{}^{e}+[F,F]_{\bar{2}b}{}^{e}f_{e}{}^{a}-[F,F]_{\bar{2}\bar{1}}{}^{a}u_{b}+[F,F]_{\bar{2}b}{}^{\bar{1}}U^{a}\\&\quad+[F,F]_{\bar{2}b}{}^{\bar{2}}V^{a}-[F,F]_{\bar{2}\bar{3}}{}^{a}w_{b}+[F,F]_{\bar{2}b}{}^{\bar{3}}W^{a}=0,\end{aligned}$$

$(2.5)_{10}$

$$\begin{aligned}&[F,F]_{\bar{3}e}{}^{a}f_{b}{}^{e}+[F,F]_{\bar{3}b}{}^{e}f_{e}{}^{a}-[F,F]_{\bar{3}\bar{1}}{}^{a}u_{b}+[F,F]_{\bar{3}b}{}^{\bar{1}}U^{a}\\&\quad-[F,F]_{\bar{3}\bar{2}}{}^{a}v_{b}+[F,F]_{\bar{3}b}{}^{\bar{2}}V^{a}+[F,F]_{\bar{3}b}{}^{\bar{3}}W^{a}=0,\end{aligned}$$

$(2.5)_{11}$

$$\begin{aligned}&[F,F]_{\bar{1}e}{}^{a}V^{e}+[F,F]_{\bar{1}\bar{2}}{}^{e}f_{e}{}^{a}+[F,F]_{\bar{1}\bar{2}}{}^{\bar{1}}U^{a}+[F,F]_{\bar{1}\bar{2}}{}^{\bar{2}}V^{a}\\&\quad-\alpha[F,F]_{\bar{1}\bar{3}}{}^{a}+[F,F]_{\bar{1}\bar{2}}{}^{\bar{3}}W^{a}=0,\end{aligned}$$

$(2.5)_{12}$

$$\begin{aligned}&[F,F]_{\bar{1}e}{}^{a}W^{e}+[F,F]_{\bar{1}\bar{3}}{}^{e}f_{e}{}^{a}+[F,F]_{\bar{1}\bar{3}}{}^{\bar{1}}U^{a}+\alpha[F,F]_{\bar{1}\bar{2}}{}^{a}\\&\quad+[F,F]_{\bar{1}\bar{3}}{}^{\bar{2}}V^{a}+[F,F]_{13}{}^{\bar{3}}W^{a}=0,\end{aligned}$$

$(2.5)_{13}$

$$\begin{aligned}&[F,F]_{\bar{2}e}{}^{a}U^{e}+[F,F]_{\bar{2}\bar{1}}{}^{e}f_{e}{}^{a}+[F,F]_{\bar{2}\bar{1}}{}^{\bar{1}}U^{a}+[F,F]_{\bar{2}\bar{1}}{}^{\bar{2}}V^{a}-\beta[F,F]_{\bar{2}\bar{3}}{}^{a}\\&\quad-[F,F]_{\bar{2}\bar{1}}{}^{\bar{3}}W^{a}=0,\end{aligned}$$

$$(2.5)_{14}\qquad [F,F]_{\bar{2}e}{}^{a}W^{e}+[F,F]_{\bar{2}\bar{3}}{}^{e}f_{e}^{a}+\beta[F,F]_{\bar{2}\bar{1}}{}^{a}+[F,F]_{\bar{2}\bar{3}}{}^{\bar{1}}U^{a}+[F,F]_{\bar{2}\bar{3}}{}^{\bar{2}}V^{a}+[F,F]_{\bar{2}\bar{3}}{}^{\bar{3}}W^{a}=0,$$

$$(2.5)_{15}\qquad [F,F]_{\bar{3}e}{}^{a}U^{e}+[F,F]_{\bar{3}\bar{1}}{}^{e}f_{e}{}^{a}+[F,F]_{\bar{3}\bar{1}}{}^{\bar{1}}U^{a}+\gamma[F,F]_{\bar{3}\bar{2}}{}^{a}+[F,F]_{\bar{3}\bar{1}}{}^{\bar{2}}V^{a}+[F,F]_{\bar{3}\bar{1}}{}^{\bar{3}}W^{a}=0,$$

$$(2.5)_{16}\qquad [F,F]_{\bar{3}e}{}^{a}V^{e}+[F,F]_{\bar{3}\bar{2}}{}^{e}f_{e}{}^{a}-\gamma[F,F]_{\bar{3}\bar{1}}{}^{a}+[F,F]_{\bar{3}\bar{2}}{}^{\bar{1}}U^{a}+[F,F]_{\bar{3}\bar{2}}{}^{\bar{2}}V^{a}+[F,F]_{\bar{3}\bar{2}}{}^{\bar{3}}W^{a}=0.$$

We now assume that $[F,F]_{cb}{}^{a}$, $[F,F]_{cb}{}^{\bar{1}}$, $[F,F]_{cb}{}^{\bar{2}}$, $[F,F]_{cb}{}^{\bar{3}}$ are all vanish on M. Under these conditions $(2.5)_1$ becomes

$$(2.6)\qquad [F,F]_{c\bar{1}}{}^{a}u_{b}+[F,F]_{c\bar{2}}{}^{a}v_{b}+[F,F]_{c\bar{3}}{}^{a}w_{b}=0.$$

Transvecting (2.6) with U^b, V^b and W^b respectively, we find

$$(2.7)_1\qquad (1-\beta^2-\gamma^2)[F,F]_{c\bar{1}}{}^{a}-\alpha\beta[F,F]_{c\bar{2}}{}^{a}-\alpha\gamma[F,F]_{c\bar{3}}{}^{a}=0,$$

$$(2.7)_2\qquad -\alpha\beta[F,F]_{c\bar{1}}{}^{a}+(1-\alpha^2-\gamma^2)[F,F]_{c\bar{2}}{}^{a}+\beta\gamma[F,F]_{c\bar{3}}{}^{a}=0,$$

$$(2.7)_3\qquad -\alpha\gamma[F,F]_{c\bar{1}}{}^{a}+\beta\gamma[F,F]_{c\bar{2}}{}^{a}+(1-\alpha^2-\beta^2)[F,F]_{c\bar{3}}{}^{a}=0.$$

If $1-\alpha^2-\beta^2-\gamma^2$ is almost everywhere non-zero on M, then, from $(2.7)_1$, $(2.7)_2$ and $(2.7)_3$, we have

$$(2.8)\qquad [F,F]_{c\bar{1}}{}^{a}=0,\qquad [F,F]_{c\bar{2}}{}^{a}=0,\qquad [F,F]_{c\bar{3}}{}^{a}=0$$

because of

$$\det\begin{pmatrix}1-\beta^2-\gamma^2 & -\alpha\beta & -\alpha\gamma\\ -\alpha\beta & 1-\alpha^2-\gamma^2 & \beta\gamma\\ -\alpha\gamma & \beta\gamma & 1-\alpha^2-\beta^2\end{pmatrix}=(1-\alpha^2-\beta^2-\gamma^2)^2.$$

Transvecting $(2.5)_2\sim(2.5)_{16}$ with U^b, V^b, W^b respectively and taking account of $[F,F]_{cb}{}^{a}=0$, $[F,F]_{cb}{}^{\bar{s}}=0$ $(s=1,2,3)$ and (2.8), we also find

$$(2.9)\qquad \begin{cases}[F,F]_{\bar{1}\bar{2}}{}^{a}=0, & [F,F]_{\bar{1}\bar{3}}{}^{a}=0, & [F,F]_{\bar{2}\bar{3}}{}^{a}=0,\\ [F,F]_{c\bar{1}}{}^{\bar{1}}=0, & [F,F]_{c\bar{1}}{}^{\bar{2}}=0, & [F,F]_{c\bar{1}}{}^{\bar{3}}=0,\\ [F,F]_{c\bar{2}}{}^{\bar{1}}=0, & [F,F]_{c\bar{2}}{}^{\bar{2}}=0, & [F,F]_{c\bar{2}}{}^{\bar{3}}=0,\\ [F,F]_{c\bar{3}}{}^{\bar{1}}=0, & [F,F]_{c\bar{3}}{}^{\bar{2}}=0, & [F,F]_{c\bar{3}}{}^{\bar{3}}=0,\end{cases}$$

$$\begin{cases} [F,F]_{12}^{1}=0, & [F,F]_{12}^{2}=0, & [F,F]_{12}^{3}=0, \\ [F,F]_{13}^{1}=0, & [F,F]_{13}^{2}=0, & [F,F]_{13}^{3}=0, \\ [F,F]_{23}^{1}=0, & [F,F]_{23}^{2}=0, & [F,F]_{23}^{3}=0. \end{cases}$$

Similarly, we can prove that if the function $1-\alpha^2-\beta^2-\gamma^2$ is non-zero almost everywhere on M and

$[F,F]_{cb}{}^{a}=0$, $[F,F]_{c1}{}^{a}=0$, $[F,F]_{c2}{}^{a}=0$, $[F,F]_{c3}{}^{a}=0$, then the other twenty-four components of the Nijenhuis tensor $(2.3)_2$, $(2.3)_3$, $(2.3)_4$ and $(2.3)_8\sim(2.3)_{28}$ are all zero on M. Hence, we have

LEMMA 2.2. *If the function* $1-\alpha^2-\beta^2-\gamma^2$ *is almost everywhere non-zero and* $[F,F]_{cb}{}^{a}$ *and* $[F,F]_{cb}{}^{s}$ *(or* $[F,F]_{cb}{}^{a}$ *and* $[F,F]_{cs}{}^{a}$*)* $(s=1,2,3)$ *are identically vanish on M, then the other twenty-four components of the Nijenhuis tensor* $(2.3)_5$, $(2.3)_6$, $(2.3)_7$ *and* $(2.3)_8\sim(2.3)_{28}$ *(or* $(2.3)_2$, $(2.3)_3$, $(2.3)_4$ *and* $(2.3)_8\sim(2.3)_{28}$*) are all vanish.*

If a symmetric affine connection ∇ is given on M, then we can easily see that the components $[F,F]_{cb}{}^{a}$, $[F,F]_{cb}{}^{1}$, $[F,F]_{cb}{}^{2}$ and $[F,F]_{cb}{}^{3}$ can be written as follows;

$$[F,F]_{cb}{}^{a}=f_c{}^e\nabla_e f_b{}^a-f_b{}^e\nabla_e f_c{}^a-(\nabla_c f_b{}^e-\nabla_b f_c{}^e)f_e{}^a+(\nabla_c u_b-\nabla_b u_c)U^a +(\nabla_c v_b-\nabla_b v_c)V^a+(\nabla_c w_b-\nabla_b w_c)W^a, \tag{2.10}$$

$$[F,F]_{cb}{}^{1}=-f_c{}^e\nabla_e u_b+f_b{}^e\nabla_e u_c+(\nabla_c f_b{}^e-\nabla_b f_c{}^e)u_e-\gamma(\nabla_c v_b-\nabla_b v_c) +\beta(\nabla_c w_b-\nabla_b w_c), \tag{2.11}$$

$$[F,F]_{cb}{}^{2}=-f_c{}^e\nabla_e v_b+f_b{}^e\nabla_e v_c+(\nabla_c f_b{}^e-\nabla_b f_c{}^e)v_e+\gamma(\nabla_c u_b-\nabla_b u_c) +\alpha(\nabla_c w_b-\nabla_b w_c), \tag{2.12}$$

$$[F,F]_{cb}{}^{3}=-f_c{}^e\nabla_e w_b+f_b{}^e\nabla_e w_c+(\nabla_c f_b{}^e-\nabla_b f_c{}^e)w_e-\beta(\nabla_c u_b-\nabla_b u_c) -\alpha(\nabla_c v_b-\nabla_b v_c), \tag{2.13}$$

that is, we find that all the partial differentiations ∂_e involved in $[F,F]_{cb}{}^{a}$, $[F,F]_{cb}{}^{1}$, $[F,F]_{cb}{}^{2}$, $[F,F]_{cb}{}^{3}$ can be replaced by the covariant differentiations ∇_e.

Thus we have

LEMMA 2.3. *If M is a differentiable manifold with $(f, g, u_{(k)}, \alpha_{(k)})$-structure, then the sets of components* $[F,F]_{cb}{}^{a}$, $[F,F]_{cb}{}^{\bar{1}}$, $[F,F]_{cb}{}^{\bar{2}}$, $[F,F]_{cb}{}^{\bar{3}}$, $[F,F]_{c\bar{1}}{}^{a}$, $[F,F]_{c\bar{2}}{}^{a}$, $[F,F]_{c\bar{3}}{}^{a}$, $[F,F]_{\bar{1}\bar{2}}{}^{a}$, $[F,F]_{\bar{1}\bar{3}}{}^{a}$, $[F,F]_{\bar{2}\bar{3}}{}^{a}$, $[F,F]_{c\bar{1}}{}^{\bar{1}}$, $[F,F]_{c\bar{1}}{}^{\bar{2}}$, $[F,F]_{c\bar{1}}{}^{\bar{3}}$, $[F,F]_{c\bar{2}}{}^{\bar{1}}$, $[F,F]_{c\bar{2}}{}^{\bar{2}}$, $[F,F]_{c\bar{2}}{}^{\bar{3}}$, $[F,F]_{c\bar{3}}{}^{\bar{1}}$, $[F,F]_{c\bar{3}}{}^{\bar{2}}$, $[F,F]_{c\bar{3}}{}^{\bar{3}}$, $[F,F]_{\bar{1}\bar{2}}{}^{\bar{1}}$, $[F,F]_{\bar{1}\bar{2}}{}^{\bar{2}}$, $[F,F]_{\bar{1}\bar{2}}{}^{\bar{3}}$, $[F,F]_{\bar{1}\bar{3}}{}^{\bar{1}}$, $[F,F]_{\bar{1}\bar{3}}{}^{\bar{2}}$, $[F,F]_{\bar{1}\bar{3}}{}^{\bar{3}}$, $[F,F]_{\bar{2}\bar{3}}{}^{\bar{1}}$, $[F,F]_{\bar{2}\bar{3}}{}^{\bar{2}}$ *and* $[F,F]_{\bar{2}\bar{3}}{}^{\bar{3}}$ *of the Nijenhuis tensor of the almost complex structure F on $M\times R^3$ define twenty-eight tensor fields in the manifold M, which are determined by the given $(f, g, u_{(k)}, \alpha_{(k)})$-structure.*

We can get directly from Lemma 2.2

PROPOSITION 2.4. *The almost complex structure F on $M\times R^3$ is integrable if and only if the four tensors* $[F,F]_{cb}{}^{a}$, $[F,F]_{cb}{}^{\bar{1}}$, $[F,F]_{cb}{}^{\bar{2}}$ *and* $[F,F]_{cb}{}^{\bar{3}}$ *vanish identically on M, or, if and only if the four tensors* $[F,F]_{cb}{}^{a}$, $[F,F]_{c\bar{1}}{}^{a}$, $[F,F]_{c\bar{2}}{}^{a}$ *and* $[F,F]_{c\bar{3}}{}^{a}$ *vanish identically on M, provided that $1-\alpha^2-\beta^2-\gamma^2$ is non-zero almost everywhere.*

§ 3. A Riemannian metric on $M\times R^3$

Let M be a differentiable manifold with $(f, g, u_{(k)}, \alpha_{(k)})$-structure. If we consider a Riemannian metric G on $M\times R^3$ with components

$$(3.1)\qquad (G_{CB})=\begin{pmatrix} g_{cb} & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix},$$

g_{cb} being the components of Riemannian metric g on M, then we see that (F,G) defines an almost Hermitian structure on $M\times R^3$, F having the almost complex structure (2.1), that is,

$$(3.2)\qquad G_{CB}F_D{}^{C}F_E{}^{B}=G_{DE},$$

where $F_C{}^B$ are components of F.

We denote the Christoffel symbols formed with G and these formed with g respectively by $\widetilde{\left\{ {A \atop C\,B} \right\}}$ and by $\left\{ {a \atop c\,b} \right\}$. Then we find that

$$\widetilde{\left\{ {a \atop c\,b} \right\}} = \left\{ {a \atop c\,b} \right\}, \tag{3.3}$$

$$\left\{ {\bar{1} \atop c\,b} \right\} = \widetilde{\left\{ {\bar{2} \atop c\,b} \right\}} = \widetilde{\left\{ {\bar{3} \atop \bar{1}\,b} \right\}} = \widetilde{\left\{ {a \atop \bar{1}\,b} \right\}} = \widetilde{\left\{ {a \atop \bar{2}\,b} \right\}} = \cdots = \widetilde{\left\{ {\bar{3} \atop \bar{3}\,\bar{3}} \right\}} = 0. \tag{3.4}$$

We denote by ∇_B and ∇_b the covariant differentiations with respect to $\widetilde{\left\{ {A \atop C\,B} \right\}}$ and $\left\{ {a \atop c\,b} \right\}$ respectively. Then the covariant derivative of F with respect to $\widetilde{\left\{ {A \atop C\,B} \right\}}$ is given by

$$\widetilde{\nabla}_C F_B{}^A = \partial_C F_B{}^A - \widetilde{\left\{ {E \atop C\,B} \right\}} F_E{}^A + \widetilde{\left\{ {A \atop C\,E} \right\}} F_E{}^B, \tag{3.5}$$

that is, given by

$$\widetilde{\nabla}_c F_b{}^a = \partial_c f_b{}^a - \left\{ {e \atop c\,b} \right\} f_e{}^a + \left\{ {a \atop c\,e} \right\} f_b{}^e = \nabla_c f_b{}^a, \tag{3.6$_1$}$$

$$\widetilde{\nabla}_c F_b{}^{\bar{1}} = -\nabla_c u_b, \quad \widetilde{\nabla}_c F_b{}^{\bar{2}} = -\nabla_c v_b, \quad \widetilde{\nabla}_c F_b{}^{\bar{3}} = -\nabla_c w_b, \tag{3.6$_2$}$$

$$\widetilde{\nabla}_c F_{\bar{1}}{}^a = \nabla_c U^a, \quad \widetilde{\nabla}_c F_{\bar{2}}{}^a = \nabla_c V^a, \quad \widetilde{\nabla}_c F_{\bar{3}}{}^a = \nabla_c W^a, \tag{3.6$_3$}$$

$$\begin{aligned} \widetilde{\nabla}_{\bar{1}} F_b{}^a &= \widetilde{\nabla}_{\bar{2}} F_b{}^a = \widetilde{\nabla}_{\bar{3}} F_b{}^a = \widetilde{\nabla}_{\bar{1}} F_b{}^{\bar{1}} = \widetilde{\nabla}_{\bar{1}} F_b{}^{\bar{2}} = \widetilde{\nabla}_{\bar{1}} F_b{}^{\bar{3}} = \widetilde{\nabla}_{\bar{2}} F_b{}^{\bar{1}} = \widetilde{\nabla}_{\bar{2}} F_b{}^{\bar{2}} \\ &= \widetilde{\nabla}_{\bar{2}} F_b{}^{\bar{3}} = \widetilde{\nabla}_{\bar{3}} F_b{}^{\bar{1}} = \widetilde{\nabla}_{\bar{3}} F_b{}^{\bar{2}} = \widetilde{\nabla}_{\bar{3}} F_b{}^{\bar{3}} = \widetilde{\nabla}_{\bar{1}} F_{\bar{1}}{}^a = \widetilde{\nabla}_{\bar{1}} F_{\bar{2}}{}^a = \widetilde{\nabla}_{\bar{1}} F_{\bar{3}}{}^a \\ &= \widetilde{\nabla}_{\bar{2}} F_{\bar{1}}{}^a = \widetilde{\nabla}_{\bar{2}} F_{\bar{2}}{}^a = \widetilde{\nabla}_{\bar{2}} F_{\bar{3}}{}^a = \widetilde{\nabla}_{\bar{3}} F_{\bar{1}}{}^a = \widetilde{\nabla}_{\bar{3}} F_{\bar{2}}{}^a = \widetilde{\nabla}_{\bar{3}} F_{\bar{3}}{}^a = \widetilde{\nabla}_{\bar{1}} F_{\bar{1}}{}^{\bar{2}} \\ &= \cdots = \widetilde{\nabla}_{\bar{3}} F_{\bar{3}}{}^{\bar{2}} = 0, \end{aligned} \tag{3.6$_4$}$$

$$\begin{aligned} &\widetilde{\nabla}_c F_{\bar{1}}{}^{\bar{2}} = -\nabla_c \gamma, \quad \widetilde{\nabla}_c F_{\bar{1}}{}^{\bar{3}} = -\nabla_c \beta, \quad \widetilde{\nabla}_c F_{\bar{2}}{}^{\bar{1}} = \nabla_c \gamma, \quad \widetilde{\nabla}_c F_{\bar{2}}{}^{\bar{3}} = -\nabla_c \alpha, \\ &\widetilde{\nabla}_c F_{\bar{3}}{}^{\bar{2}} = \nabla_c \alpha, \quad \widetilde{\nabla}_c F_{\bar{3}}{}^{\bar{1}} = \nabla_c \beta. \end{aligned} \tag{3.6$_5$}$$

Hence we have

PROPOSITION 3.1. *Suppose that M has an $(f,\ g,\ u_{(k)},\ \alpha_{(k)})$-structure. Then a necessary and sufficient condition for the product Riemannian manifold $M\times R^3$ to be a Kählerian space with (F, G) is that all of $f, u, v, w, \alpha, \beta$ and γ are covariantly constant on M.*

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