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ON AN INTEGRAL OF POWERS OF A SPIRALLIKE FUNCTION

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1. Introduction

Let S denote the class of analytic, univalent (one-to-one) functions f in $E=\{z : |z|<1\}$ that are normalized by $f(0)=0$ and $f'(0)=1$. The set of spirallike functions S_p is the subclass of S consisting of functions f for which there exists a real β, $|\beta|<\pi/2$, such that $\mathrm{Re}\{e^{i\beta}zf'(z)/f(z)\}>0$, $z\in E$. The starlike functions S^* is the subset of spirallike functions for which the constant β can be taken to be zero.

For $f\in S$ and for a complex number α, define

$$g_\alpha(z)=\int_0^z [f(\zeta)/\zeta]^\alpha d\zeta. \tag{1}$$

A number of papers have appeared ([2], [3], [6]) that determine choices of α such that $g_\alpha\in S$ whenever $f\in S$. It is not difficult to prove, by normal family arguments, that the set A of complex numbers α for which $g_\alpha\in S$ whenever $f\in S$ is closed. The determination of the boundary of A, however, or even other properties of the set A such as its connectedness, appear to be quite difficult. In this paper, we consider the set A_p of complex numbers α for which $g_\alpha\in S$ whenever $f\in S_p$ and determine two closed sets I_p and O_p such that $I_p\subset A_p\subset O_p$. These sets are improvements of results of Causey [2]. Clearly $A\subset A_p$.

2. The set O_p

Royster [7] established the following lemma for an analogous problem to the one treated here.

LEMMA 1. *The function* $g(z)=\exp[\mu\log(1+z)]$ *is univalent in* E *if and only if* $\mu\neq 0$ *lies in one of the closed disks* $|\mu+1|\leq 1$, $|\mu-1|\leq 1$.

The function $g(z)$ in this lemma, after normalization and definition of the parameter μ, plays the role of the function g_α in (1) provided the integrand is suitably defined. In order to determine when this integrand is in S_p, we establish the following result.

LEMMA 2. *The function* $f(z)=z\exp[\mu\log(1+z)]$ *is univalent (and spirallike) in* E *if and only if* $|\mu+1|\leq 1$.

PROOF. Set

$$F(z)=\frac{zf'(z)}{f(z)}=\frac{1+(1+\mu)z}{1+z}.$$

Case 1. $|\mu+1|>1$. In this case $f'(z)$ has a zero at $z=-1/(\mu+1)$ which is in E. Hence $f(z)$ is not univalent in E.

Case 2. $|\mu+1|<1$. The linear fractional transformation $w=F(z)$ maps the unit circle $U=\{z:|z|=1\}$ onto a straight line that has one and only one point in common with the real axis and this point is in the interval $0<z<1$. Indeed, $F(-1)=\infty$ so $F(U)$ is a straight line. The interior point 0 of E and the exterior point $-1/(\mu+1)$ of E are mapped by F respectively to 1 and 0. Therefore the line segment joining 0 and 1 in the w-plane must cross the line $F(U)$. It follows that there is a real β, $|\beta|<\pi/2$, such that

$$\mathrm{Re}\left\{e^{i\beta}\frac{zf'(z)}{f(z)}\right\}=\mathrm{Re}\{e^{i\beta}F(z)\}>0,\ z\in E.$$

Since this implies f is univalent [8], we conclude $f\in S_p$.

Case 3. $|\mu+1|=1$, $\mu\neq 0$. The image of the unit circle U by $w=F(z)$ is a straight line through the origin which, since $F(0)=1$, is not the real axis. We conclude, as in the previous case, that $f\in S_p$.

THEOREM 1. *For each complex number* α *in* $|\alpha|>1/2$ *there is a spirallike function* f *such that* $g_\alpha\notin S$ *where* g_α *is defined by* (1).

PROOF. Let $f(z)=z\exp[\mu\log(1+z)]$ where $|\mu+1|\leq 1$. By Lemma 2, $f\in S_p$ and, for complex α,

$$g_\alpha(z)=\int_0^z\exp[\alpha\mu\ \log(1+\zeta)]\,d\zeta=\frac{1}{\alpha\mu+1}\{\exp[(\alpha\mu+1)\log(1+z)-1\} \tag{2}$$

provided $\alpha\mu\neq -1$. The constants in (2) are immaterial as far as the univalence of g_α is concerned. Hence, by Lemma 1, g_α is univalent if and only if $\omega\neq -1$ and $|\omega|\leq 1$ or $|\omega+2|\leq 1$, where $\omega=\alpha\mu$. Now the disk $|\omega+\alpha|\leq|\alpha|$, which for $\alpha\neq 0$ is the same set as in the hypothesis $|\mu+1|\leq 1$, is contained in $|\omega|\leq 1$ if and only if $|\alpha|\leq 1/2$. For $|\alpha|>1/2$, however, there is always a point in $|\omega+\alpha|\leq|\alpha|$ that is in the exterior of $|\omega+2|\leq 1$ and $|\omega|\leq 1$. This implies there is a choice of μ such that the function g_α in (1) is not univalent in E. If $\omega=-1$, then

$g_\alpha = \log(1+z)$ which is in S.

Causey [2] proved the special case of Theorem 1 for which α is real and $\alpha > 1/2$.

When $\mu = -2$ in Lemma 1, $f(z) = z/(1+z)^2$ in S^*. The argument used to establish Theorem 1 when applied to this special case yields the following.

THEOREM 2. *The function*

$$g_\alpha(z) = \int_0^z \frac{d\zeta}{(1+\zeta)^{2\alpha}}$$

is univalent in E if and only if $|\alpha| \leq 1/2$ or $|\alpha - 1| \leq 1/2$.

This theorem proves that the set of points A^* in the α-plane such that g_α in (1) is a member of S whenever $f \in S^*$ is contained in the closed set $|\alpha| \leq 1/2$, $|\alpha - 1| \leq 1/2$. Merkes and Wright [5] have shown that A^* contains the real interval $-1/2 \leq \alpha \leq 3/2$ contained in this set. One consequence of the theorem in the next section of this paper is that there are nonreal points in A.

3. The set I_p

Causey [3] and Kim [4] showed that A contains the disk $|\alpha| \leq (\sqrt{2}-1)/4 \approx .1035$ and $.1103$, respectively. These results were improved in the next theorem.

THEOREM 3. *If $f \in S$, then g_α in (1) is in S for $|\alpha| \leq 1/4$.*

PROOF. We have for $|z| = r < 1$ that

$$\left|\frac{zg_\alpha''}{g_\alpha'}\right| = |\alpha| \left|\frac{zf'}{f} - 1\right| \leq |\alpha| \left\{\left|\frac{zf}{f}\right| + 1\right\} \leq |\alpha| \left(\frac{1+r}{1-r} + 1\right) \leq \frac{2|\alpha|(1+r)}{1-r^2} \leq \frac{4|\alpha|}{1-r^2}$$

Now Becker in his thesis [1] has proved that an analytic function g in E that satisfies $|zg''/g'| \leq 1/(1-|z|^2)$ in E is univalent in the unit disk. This condition is satisfied by g_α provided $4|\alpha| \leq 1$ and, hence, g_α is univalent in $|\alpha| \leq 1/4$.

By Theorem 1 together with the above theorem, we conclude that

$$\{\alpha : |\alpha| \leq 1/4\} \subset A \subset A_p \subset \{\alpha : |\alpha| \leq 1/2\}.$$

Furthermore, $A_p \subset A^*$ and by Theorem 2,

$$A^* \subset \{\alpha : |\alpha| \leq 1/2 \text{ or } |\alpha - 1| \leq 1/2\}.$$

We showed that $g_\alpha \in S$ whenever $f \in S$ provided $|\alpha| \leq 1/4$. However, this bound 1/4 is not the best possible constant probably.

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