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ON QUASI-COMPLETE SPACES

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G. D. Creede [1] introduced quasi-complete spaces in connection with problems of metrizability and raised the open question whether all quasi-complete spaces are $w\Delta$-spaces. In [2], example 1.11 exhibits a p-space which is not $w\Delta$-space. Since p-spaces are quasi-complete spaces, this shows that a quasi-complete need not be $w\Delta$-space.

In this paper, we give a simple characterization of quasi-complete space, and we show that $w\Delta$-spaces and quasi-complete spaces are equivalent in the class of a completely regular θ-refinable spaces.

All topological spaces are assumed to be T_1. The set of positive integers is denoted by N.

DEFINITION 1. [1]. A T_1-space X is said to be *quasi-complete* provided that there is a sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of X with the following property; if $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of non-empty closed subsets of X and if there exists an element $x_0 \in X$ such that, for each n, there is a $B_n \in \mathcal{G}_n$ with $A_n \cup \{x_0\} \subset B_n$, then $\bigcap_{n=1}^{\infty} A_n \neq \phi$.

DEFINITION 2. [3]. A T_1-space X is a *$w\Delta$-space* if there exists a sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of open covers of X such that, if $\{A_n\}_{n=1}^{\infty}$ is a decreasing sequence of non-empty closed subsets of X and there exists $x_0 \in X$ for which $A_n \subset St\,(x_0, \mathcal{G}_n)$ for all n, then $\bigcap_{n=1}^{\infty} A_n \neq \phi$.

A sequence $\{A_n(x)\}_{n=1}^{\infty}$ of subsets of X, with $x \in A_n(x)$ for each $n \in N$, is called an *x-sequence* if $x_n \in A_n(x)$ implies that $\{x_n\}_{n=1}^{\infty}$ has a cluster point in X.

We shall give a simple characterization of quasi-complete which will be used frequently.

THEOREM 1. *X is quasi-complete if and only if there exists a sequence $\{\mathscr{G}_n\}_{n=1}^{\infty}$ of open covers of X such that, for each $x\in X$ and $B_n\in\mathscr{G}_n$ with $x\in B_n$, a sequence $\{\cap_{n=1}^{k}B_n | k\in N\}$ is an x-sequence.*

PROOF. Let $x\in X$ and $B_n\in\mathscr{G}_n$ with $x\in B_n$. For each $k\in N$, take $x_k\in\cap_{n=1}^{k}B_n$. If $\{x_k\}_{k=1}^{\infty}$ has no cluster point in X, we let $A_k=\{x_n | n\geq k\}$, then $\{A_k\}_{k=1}^{\infty}$is a decreasing sequence of non-empty closed subsets of X and $A_k\cup\{x\}\subset B_k$. Since X is quasi-complete, we have $\cap_{k=1}^{\infty}A_k\neq\phi$. But $\cap_{k=1}^{\infty}A_k=\phi$, since $\{x_k\}$ has no cluster point in X.

To prove the converse, let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of non-empty closed subsets of X and there exist $x\in X$ such that, for each n, $B_n\in\mathscr{G}_n$ with $A_n\cup\{x\}\subset B_n$. Assume $\cap_{n=1}^{\infty}A_n=\phi$. Since $A_k\neq\phi$, we can find $x_k\in A_k$ for each k. Then $x_k\in A_k\cup\{x\}\subset B_k$. Therefore $x_k\in\cap_{n=1}^{k}B_n$ for each k, since $\{A_k\}_{k=1}^{\infty}$ is a decreasing sequence. We show that $\{x_k\}$ has no cluster point in X. For each $p\in X$, since $\cap_{n=1}^{\infty}A_n=\phi$ and $A_n\supset A_{n+1}$, there exists integer N such that $p\notin A_n$ for any $n\geq N$. Then $\mathscr{C}A_N$ is open set containing p and $x_n\notin\mathscr{C}A_n$ for any $n\geq N$. Thus p is not cluster point of sequence $\{x_k\}$.

The following lemma was proved in [2].

LEMMA 2. *A completely regular space X is a p-space if and only if there is a sequence $\{\mathscr{G}_n\}_{n=1}^{\infty}$ of open covers of X satisfying;*

(*a*) *$\cap_{n=1}^{\infty}\bar{G}_n$ is compact.*

(*b*) *$\{\cap_{n=1}^{k}\bar{G} | k\in N\}$ is an x-sequence.*

Using a proof analogous to one given by Burke for Theorem 1.4 of [2], we have the following.

LEMMA 3. *A completely regular quasi-complete space X is a p-space if every closed countably compact subset of X is compact.*

PROOF. Let $\{\mathscr{G}_n\}_{n=1}^{\infty}$ be a sequence of open covers of X such that, for each

$x\in X$ and $B_n\in\mathcal{G}_n$ with $x\in B_n$, sequence $\{\bigcap_{n=1}^k B_n | k\in N\}$ is an x-sequence. Since X is completely regular space, for each n, let $\mathcal{U}_n$ be open cover of X such that $\{\bar{G} | G\in\mathcal{U}_n\}$ refines $\mathcal{G}_n$. We show that $\{\mathcal{U}_n\}$ is a sequence of open covers of X satisfying the condition of Lemma 2. Let $x\in X$ and $G_n\in\mathcal{U}_n$ with $x\in G_n$, we find $B_n\in\mathcal{G}_n$ such that $\bar{G}_n\subset B_n$. So $\bigcap_{n=1}^k \bar{G}_n\subset\bigcap_{n=1}^k B_n$. Thus $\{\bigcap_{n=1}^k \bar{G}_n | k\in N\}$ is x-sequence. Let $\{x_k\}_{k=1}^\infty$ be any sequence in $\bigcap_{n=1}^\infty \bar{G}_n$, then $x_k\in\bigcap_{n=1}^k \bar{G}_n$ for each k. Since $\{\bigcap_{n=1}^k \bar{G}_n | k\in N\}$ is x-sequence, $\{x_k\}$ has a cluster point in $\bigcap_{n=1}^\infty \bar{G}_n$. So $\bigcap_{n=1}^\infty \bar{G}_n$ is closed countable compact. By hypothesis and Lemma 2, X is p-space.

A space X is said to be *θ-refinable* [4] if, for every open covering $\mathcal{U}$ of X, there is a sequence $\{\mathcal{U}_n\}_{n=1}^\infty$ of open refinements of $\mathcal{U}$ such that, if $x\in X$, there is $m(x)\in N$ such that x is in at most a finite number of elements of $\mathcal{U}_{m(x)}$.

Burke [2] showed that p-spaces, $w\Delta$-spaces and strict p-spaces are equivalent in the class of completely regular θ-refinable spaces.

Moreover we can add quasi-complete among the above equivalent conditions.

THEOREM 4. *For a completely regular θ-refinable space X, the following conditions are equivalent*;

(1) *X is p-space,*
(2) *X is strict p-space,*
(3) *X is wΔ-space,*
(4) *X is quasi-complete.*

PROOF. (4)⇒(1) is clear from Lemma 3, since closed countably compact subsets are compact in a θ-refinable space [4].

(3)⇒(4) is clear from definition.

THEOREM 5. *X_i is quasi-complete if and only if $\prod_{i=1}^\infty X_i$ is quasi-complete.*

PROOF. if; It Is clear from the fact that closed subspace of quasi-complete is quasi-complete.

Only if; For each i, let $\{\mathcal{G}_n^i\}$ be a sequence of open covers of X_i such that, for each $x\in X_i$ and $B_n^i\in\mathcal{G}_n^i$ with $x\in B_n^i$, sequence $\{\bigcap_{n=1}^k B_n^i | k\in N\}$ is x-sequence.

Let $\mathcal{U}_1=\{<G_1^1> | G_1^1\in\mathcal{G}_1^1\}$, $\mathcal{U}_2=\{<G_1^2> | G_1^2\in\mathcal{G}_1^2\}$

$$\mathscr{U}_3=\{<G_2^1>|G_2^1\in\mathscr{G}_2^1\},\qquad \mathscr{U}_3=\{<G_1^3>|G_1^3\in\mathscr{G}_1^3\}$$

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Then $\{\mathscr{U}_n\}_{n=1}^{\infty}$ is a sequence of open covers of $\prod_{i=1}^{\infty}X_i$ satisfying the condition of Theorem 1.

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