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ON AN APPLICATION OF THE STEREOGRAPHIC PROJECTION TO CP^m

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§0. Introduction. Let M^n (resp. M'^n) be a Riemannian space of metric g (resp. g') and ϕ a diffeomorphism from M^n to M'^n. If ϕ maps any geodesic in M^n to a geodesic in M'^n, it is called projective. The projective curvature tensor W is preserved by any projective map ϕ, i.e. we have $W=\phi^*(W')$. For a diffeomorphism ϕ if there exists a scalar function σ such that $\phi^*(g')=e^{2\sigma}g$, we call ϕ conformal. The conformal curvature tensor C is preserved by any conformal map.

Let K^n be a Kählerian space and $\Gamma_{\mu}{}^{\lambda}{}_{\nu}$ Christoffel symbols with respect to a local coordinate $\{z^\lambda\}$. A curve c in K^n is called a holomorphically planar (or H-plane) curve if c is represented as $z^\lambda=z^\lambda(t)$ and satisfy

$$\frac{d^2z^\lambda}{dt^2}+\Gamma_{\mu}{}^{\lambda}{}_{\nu}\frac{dz^\mu}{dt}\frac{dz^\nu}{dt}=\alpha\frac{dz^\nu}{dt},$$

where α is a complex-valued function of t, [9], [11].

Consider a diffeomorphism ϕ of K^n to another K'^n. An *H-projective map* is a diffeomorphism which maps any H-plane curve to an H-plane curve. A holomorphic ϕ is H-projective if and only if there exists a self-adjoint vector ρ_λ such that

$$\phi^*(\Gamma'_{\mu}{}^{\lambda}{}_{\nu})=\Gamma_{\mu}{}^{\lambda}{}_{\nu}+\rho_\mu\delta^\lambda_\nu+\rho_\nu\delta^\lambda_\mu,$$

where $\Gamma'_{\mu}{}^{\lambda}{}_{\nu}$ mean the Christoffel symbols of K'^n.

We have known a lot of theorems about H-projective maps which correspond to ones of projective maps, [9], [11]. Especially, corresponding to W, the H-projective curvature tensor P has been shown as an invariant under H-projective maps, [11].

Now it would be natural to ask for a diffeomorphism ϕ of K^n to K'^n having the property x such that

projective: H-projective=conformal : x.

It seems that the Bochner curvature tensor K of K^n gives support to the existence of x. Because we may consider a symbolical relation

$$W : P = C : K$$

to be valid among the defining equations of these tensors, [4], [12]. Actually, some theorems for Riemannian spaces of $C=0$ have been generalized to for Kählerian spaces of $K=0$, [4], [5], [10]. Thus K would be preserved by ϕ of property x.

On the other hand, let S^n be the unit sphere in the Euclidean $(n+1)$-space E^{n+1}. If we denote by Φ the central (or stereographic) projection from S^n to an E^n (selected suitably), Φ is a projective (or conformal) map. Hence, for any proiective (or conformal) local transformation ϕ in E^n, $\Phi^{-1}\circ\phi\circ\Phi$ is projective (or conformal) on S^n.

The complex projective space CP^m is one of typical examples of Kählerian spaces, and is a quotient space of S^{2m+1} by a certain equivalence relation, [1]. Making use of the central (or stereographic) projection Φ of S^{2m+1} to E^{2m+1}, an equivalence relation can be introduced in E^{2m+1} and the induced map $\tilde{\Phi}$ is defined so that the commutativity holds in the diagramm:

$$\begin{array}{ccc} S^{2m+1} & \xrightarrow{\quad \Phi \quad} & E^{2m+1} \\ \downarrow & & \downarrow \\ CP^m = S^{2m+1}/\sim & \xrightarrow{\quad \tilde{\Phi} \quad} & E^{2m} = E^{2m+1}/\sim \end{array}$$

For the central projection Φ, we may expect $\tilde{\Phi}$ to be H-projective. $\tilde{\Phi}$ would have the property x for the stereographic Φ.

The purpose of this paper is mainly to discuss on $\tilde{\Phi}$ for the stereographic projection Φ. x is not fixed yet in this paper and still remains as a question.

Throughout the paper we shall agree with the following conventions.

(I) The ranges of indices.

$$\begin{aligned} &A, B, C, \cdots = 1, \cdots, 2m+2, \\ &a, b, c, \cdots = 1, \cdots, m+1, \\ &\lambda, \mu, \nu, \cdots = 1, \cdots, m, \\ &j, k, h, \cdots = 1, \cdots, m, 1^*, \cdots, m^*. \end{aligned}$$

(II) $\Delta = m+1, \ \Delta^* = 2m+2.$

(III) Indices with $*$.

For real coordinates— say $\{Y^A\}$, $Y^{a^*} = Y^{a+(m+1)}$,

For complex coordinates—say $\{z^\lambda\}$,

$z^{\lambda^*}=\bar{z}^{\lambda}$ (complex conjugate), $\{z^h\}=\{z^{\lambda}, \bar{z}^{\lambda}\}$.

(IV) The summation convention. For examples,

$$Y^AY^A=Y^1Y^1+\cdots+Y^{2m+2}Y^{2m+2},$$
$$z^{\lambda}dz^{\lambda^*}=z^1dz^{1^*}+\cdots+z^m dz^{m^*},$$
$$u_{\lambda}Y^{\lambda}=u_1Y^1+\cdots+u_mY^m.$$

§1. The canonical metric of CP^m. Consider the Euclidean space E^{2m+2} of dimension $2m+2$, $m\geq 1$, and we denote by $\{Y^A\}$ a fixed orthogonal coordinate system of origin O. Let $\{w^a\}$ be the complex coordinate system in E^{2m+2} associated to $\{Y^A\}$:

$$w^a=Y^a+iY^{a^*}.$$

S^{2m+1} means the unit hypersphere of center O defined by

$$Y^AY^A=w^aw^{a^*}=1.$$

Let (w^a) and (w'^a) be points on S^{2m+1}. If there exists a θ such that

$$(1.1) \qquad w'^a=e^{i\theta}w^a, \qquad 0\leq\theta\leq 2\pi,$$

then we shall say (w^a) to be equivalent to (w'^a), and represent it by $(w^a)\sim(w'^a)$. As this relation $\sim$ clearly satisfies the three conditions of equivalence relation, S^{2m+1} is classified into the set

$$CP^m=S^{2m+1}/\sim$$

of the equivalence classes. CP^m is called the complex projective space. It is an m (complex-) dimensional complex manifold with the natural structure. In fact, the natural local coordinates $\{V_b, z_b^{\lambda}\}$, $b=1, \cdots, m+1$, of CP^m is introduced as follows: For each b, U_b and V_b denote the sets given by

$$U_b=\{(w^a)\in S^{2m+1}\,|\,w^b\neq 0\}, \quad V_b=U_b/\sim,$$

and let $\{z_b{}^{\lambda}\}$ on V_b

$$z_b{}^{\lambda}=\frac{w^{\lambda}}{w^b}, \qquad \lambda=1, \cdots, b\text{-}1,$$

$$z_b{}^{\lambda}=\frac{w^{\lambda+1}}{w^b}, \qquad \lambda=b, \cdots, m.$$

We shall consider a geometrical meaning of (1.1). In terms of $\{Y^A\}$, (1.1) is written as

$$(1.2)\qquad y'^{a}=y^{a}\cos\theta-y^{a^*}\sin\theta,$$
$$y'^{a^*}=y^{a^*}\cos\theta+y^{a}\sin\theta.$$

Let Y and $\tilde{Y}$ be the vectors in E^{2m+2} defined by

$$Y=\begin{pmatrix} y^{a} \\ y^{a^*} \end{pmatrix}, \qquad \tilde{Y}=JY=\begin{pmatrix} -y^{a^*} \\ y^{a} \end{pmatrix}.$$

Here, J means the natural almost complex structure in E^{2m+2}, i.e. the matrix

$$J=\begin{pmatrix} O_m & -I_m \\ I_m & O_m \end{pmatrix},$$

O_m and I_m being the zero and the unit matrix respectively. As the vector Y at (Y^A) on S^{2m+1} is regarded as the unit normal vector to S^{2m+1} at the point, $\tilde{Y}$ is tangent to S^{2m+1} at (Y^A). The set of Y at each point (Y^A) on S^{2m+1} constitutes a vector field $\tilde{Y}$ over S^{2m+1}, and it is known that $\tilde{Y}$ is a unit Killing vector in S^{2m+1} with the natural structure of a space of constant curvature. $\tilde{Y}$ is called a Sasakian structure on S^{2m+1}. The equation (1.2) is written as

$$Y'=Y\cos\theta+\tilde{Y}\sin\theta.$$

Thus, the equivalence class of a point (Y^A) is a great circle which is an integral curve of the Sasakian structure $\tilde{Y}$ because of

$$\left(\frac{dY'}{d\theta}\right)_{\theta=0}=\tilde{Y}.$$

It is known that S^{2m+1} is a fibre bundle over CP^m with fibre S^1, called Hopf fibering.

Henceforward, our discussions will be done only in

$$U_{\Delta}=\{(w^a)\,|\,w^{\Delta}\neq 0\} \text{ and } V_{\Delta}=U_{\Delta}/\sim.$$

The canonical (Kähler) metric of CP^m is defined in V_{Δ} by

$$(1.3)\qquad ds_z^{\,2}=\frac{2}{f^4}(f^2dz^{\lambda}dz^{\lambda^*}-|z^{\lambda^*}dz^{\lambda}|^2),$$

where

$$z^{\lambda}=\frac{w^{\lambda}}{w^{\Delta}},$$

$$(1.4)\qquad f=\sqrt{1+u}\,,\qquad u=z^{\varepsilon}z^{\varepsilon^*}.$$

The metric (1.3) is written in the form

$$ds_z^{\,2}=g_{jh}dz^{j}dz^{h}=2g_{\lambda\mu^*}dz^{\lambda}dz^{\mu^*}$$

with $g_{\lambda\mu}=g_{\lambda^*\mu^*}=0$ and

$$g_{\lambda\mu^*}=\frac{1}{f^4}(f^2\delta_{\lambda\mu}-z^{\lambda^*}z^{\mu}).$$

(g^{jh}) is given by $g^{\lambda\mu}=g^{\lambda^*\mu^*}=0$ and

$$g^{\lambda\mu^*}=f^2(\delta_{\lambda\mu}+z^{\lambda}z^{\mu^*}).$$

The Christoffel symbols $\Gamma_{j\ l}^{\ h}$ are all zero except

$$\Gamma_{\mu\ \nu}^{\ \lambda}=-\frac{1}{f^2}(\delta_{\mu}^{\lambda}z^{\nu^*}+\delta_{\nu}^{\lambda}z^{\mu^*}) \tag{1.5}$$

and their complex conjugates.

The non-vanishing components of the curvature tensor $R^h{}_{jkl}$ are ones which follow by the algebraic identities about $R^h{}_{jkl}$ from

$$R^{\lambda}{}_{\mu\nu\omega^*}=-(\delta_{\mu}^{\lambda}g_{\nu\omega^*}+\delta_{\nu}^{\lambda}g_{\mu\omega^*})$$

and their complex conjugates, [12].

It is CP^m with this metric what we shall denote by CP^m in the rest of this paper. CP^m is a space of constant holomorphic curvature.

§2. The central projection $\tilde{\Phi}$. Denoting the north pole of S^{2m+1} by (y_0^A):

$$y_0^{\Delta}=1,\ \ y_0^A=0,\ \ A\neq\Delta,$$

we consider the tangent hyperplane

$$E^{2m+1}:\qquad Y^{\Delta}=1$$

of S^{2m+1} at (y_0^A). Let

$$\Phi: S^{2m+1}-S_{\Delta}^{2m}\longrightarrow E^{2m+1}$$

be the central projection, where S_{Δ}^{2m} denotes the equator $Y^{\Delta}=0$ on S^{2m+1}.

Consider a point $\tilde{P}\in CP^m$. If $\tilde{P}\in V_{\Delta}$, the equivalence class $\tilde{P}$ contains a point $P(y^A)\in S^{2m+1}$ such that $y^{\Delta}\neq 0$. As the equation of line OP is

$$Y^A=ty^A,\quad t:\text{real},$$

the coordinates of $\Phi(P)$ are

$$u^A=\frac{y^A}{y^{\Delta}}.$$

If $P'(y'^A)$ be a point equivalent to P, the coordinates of $\Phi(P')$ are $u'^A=y'^A/y'^{\Delta}$ which are written by virtue of (1.2) as

$$(2.1)\qquad \begin{aligned} u'^{\lambda} &= (y^{\lambda}\cos\theta - y^{\lambda^*}\sin\theta)/y'^{\Delta}, \\ u'^{\lambda^*} &= (y^{\lambda}\sin\theta + y^{\lambda^*}\cos\theta)/y'^{\Delta}, \\ u'^{\Delta^*} &= (y^{\Delta}\sin\theta + y^{\Delta^*}\cos\theta)/y'^{\Delta} \end{aligned}$$

on E^{2m+1}, where

$$y'^{\Delta} = y^{\Delta}\cos\theta - y^{\Delta^*}\sin\theta.$$

Since Φ is the central projection and $\tilde{P}$ is a great circle of S^{2m+1}, the equation (2.1) represents a line on E^{2m+1} with parameter θ. On the other hand, it is evident geometrically that different two points of V_{Δ} go to two non-intersecting lines on E_{2m+1}. Thus Φ induces a map from V_{Δ} into $E^{2m+1}/\sim$, where $\sim$ means the equivalence relation induced by Φ.

Next we shall consider

$$(2.2)\qquad E^{2m}: \quad Y^{\Delta^*} = 0$$

on E^{2m+1}, and find the point (denoted by $\tilde{\Phi}(\tilde{P})$) at where the line (2.1) meets with E^{2m}.

At the point, we have from (2.1) and (2.2)

$$y^{\Delta}\sin\theta + y^{\Delta^*}\cos\theta = 0.$$

Substituting these values of θ into (2.1), we can get

$$(2.3)\qquad \begin{aligned} u'^{\lambda} &= \alpha(y^{\Delta}y^{\lambda} + y^{\Delta^*}y^{\lambda^*}), \\ u'^{\lambda^*} &= \alpha(y^{\Delta}y^{\lambda^*} - y^{\Delta^*}y^{\lambda}), \end{aligned}$$

where

$$(2.4)\qquad 1/\alpha = (y^{\Delta})^2 + (y^{\Delta^*})^2 = w^{\Delta}w^{\Delta^*}.$$

Thus we obtain a map

$$\tilde{\Phi}: V_{\Delta} \longrightarrow E^{2m}$$

which brings $\tilde{P}$ to $\tilde{\Phi}(\tilde{P})$ given by (2.3).

We shall represent $\tilde{\Phi}$ in terms of local coordinates

$$(2.5)\qquad z^{\lambda} = \frac{w^{\lambda}}{w^{\Delta}} \quad \text{in } V_{\Delta}$$

and the complex coordinate in E^{2m} given by

$$(2.6)\qquad \alpha^{\lambda} = u'^{\lambda} + iu'^{\lambda^*}$$

which is the restriction of $\{w^a\}$ in E^{2m+2} to E^{2m}.

Substituting (2.3) into (2.6) and taking account of (2.4) and (2.5), we obtain $\alpha^{\lambda} = z^{\lambda}$. Hence we know that $\tilde{\Phi}$ is given by

$$z^\lambda \longrightarrow \alpha^\lambda = z^\lambda$$

and consequently $\tilde{\Phi}$ is 1-1 holomorphic.

$\tilde{\Phi}$ will be called the *central projection* of CP^m to E^{2m}.

Now consider the canonical (Kähler) metric

$$ds_\alpha{}^2 = 2d\alpha^\lambda d\alpha^{\lambda*}$$

on E^{2m}, then the induced metric on V_Δ by $\tilde{\Phi}$ is

$$\tilde{\Phi}^*(ds_\alpha{}^2) = 2\,dz^\lambda dz^{\lambda*}. \tag{2.7}$$

The Christoffel symbols $\tilde{\Phi}^*(\Gamma'_\mu{}^\lambda{}_\nu)$ of (2.7) being all zero, we have

$$\tilde{\Phi}^*(\Gamma'^\lambda_{\mu\nu}) = \Gamma_\mu{}^\lambda{}_\nu + \frac{1}{f^2}(\delta^\lambda_\mu z^{\nu*} + \delta^\lambda_\nu z^{\mu*})$$

by taking account of (1.5). Consequently, $\tilde{\Phi}$ is an H-projective transformation.

§ 3. **The stereographic projection $\tilde{\Psi}$.** Let $P_1(y_1^A)$ be the south pole of S^{2m+1} given by

$$y_1^\Delta = -1, \qquad y_1^A = 0, \qquad A \neq \Delta.$$

Consider the stereographic projection

$$\Psi : S^{2m+1} - \{P_1\} \longrightarrow E^{2m+1}$$

where

$$E^{2m+1}: \qquad Y^\Delta = 0.$$

For a point $P(y^A) \neq P_1$, the line P_1P is given by

$$Y^A = y_1^A + t(y^A - y_1^A), \qquad t : \text{real},$$

or equivalently

$$Y^A = ty^A, \qquad A \neq \Delta,$$
$$Y^\Delta = -1 + t(y^\Delta + 1).$$

Therefore the value of t is

$$t = \frac{1}{y^\Delta + 1}$$

at the point $\Psi(P)$, the intersecting point of line P_1P with E^{2m+1}.

Thus the coordinates of $\Psi(P)$ on E^{2m+1} are

$$u^A = y^A/(y^\Delta + 1), \qquad A \neq \Delta.$$

For a point $P'(y'^A)$ equivalent to P, $\tilde{\Psi}(P')$ has coordinates

$$u'^A = y'^A/(y'^\Delta + 1), \qquad A \neq \Delta$$

and hence taking account of (1.2)

$$u'^{\lambda}=\beta(y^{\lambda}\cos\theta-y^{\lambda^*}\sin\theta)$$

(3.1) $$u'^{\lambda^*}=\beta(y^{\lambda^*}\cos\theta+y^{\lambda}\sin\theta)$$

$$u'^{\Delta^*}=\beta(y^{\Delta^*}\cos\theta+y^{\Delta}\sin\theta)$$

where

(3.2) $$\frac{1}{\beta}=y'^{\Delta}+1=y^{\Delta}\cos\theta-y^{\Delta^*}\sin\theta+1.$$

As Ψ is a conformal map, the class $\tilde{P}$ of P is mapped by Ψ to a circle or a line given by (3.1).

Denoting

$$E^{2m}:\qquad Y^{\Delta^*}=0,$$

we shall find the intersecting points of (3.1) with E^{2m}. At the points θ takes the values such that

(3.3) $$y^{\Delta^*}\cos\theta+y^{\Delta}\sin\theta=0,$$

and hence we have

(3.4) $$\cos\theta=\pm y^{\Delta}/\sqrt{(y^{\Delta})^2+(y^{\Delta^*})^2}=\pm y^{\Delta}/\sqrt{w^{\Delta}\,w^{\Delta^*}}$$

if $\tilde{P}$ is in V_{Δ}.

In the following we shall adopt $+$ sign in (3.4) and denote by $\tilde{\Psi}(\tilde{P})$ the point corresponding to that value of θ.

The coordinates of $\tilde{\Psi}(\tilde{P})$ are

(3.5) $$u'^{\lambda}=\gamma(y^{\Delta}y^{\lambda}+y^{\Delta^*}y^{\lambda^*}),$$
$$u'^{\lambda^*}=\gamma(y^{\Delta}y^{\lambda^*}-y^{\Delta^*}y^{\lambda})$$

by (3.1), (3.2), (3.3) and (3.4), where

(3.6) $$\gamma=1/(w^{\Delta}w^{\Delta^*}+\sqrt{w^{\Delta}w^{\Delta^*}}).$$

Next we shall represent $\tilde{\Psi}$ in terms of local coordinates $\{z^{\lambda}\}$ in V_{Δ} and

(3.7) $$\alpha^{\lambda}=u'^{\lambda}+iu'^{\lambda^*}$$

in E^{2m}. Substituting (3.5) into (3.7) and taking account of

$$z^{\lambda}=\frac{w^{\lambda}}{w^{\Delta}}=\frac{y^{\lambda}+iy^{\lambda^*}}{y^{\Delta}+iy^{\Delta^*}}$$

and (3.6), we can get

(3.8) $$\alpha^{\lambda}=z^{\lambda}/(f+1),$$

where f means the one in (1.4), i.e.,

$$=\sqrt{1+u},\qquad u=z^{\varepsilon}\bar{z}^{\varepsilon^*}.$$

As we have

$$(3.9)\qquad \alpha^\lambda\alpha^{\lambda*}=\frac{f-1}{f+1},\qquad f=\frac{1+\alpha^\varepsilon\alpha^{\varepsilon*}}{1-\alpha^\varepsilon\alpha^{\varepsilon*}},$$

the equation (3.8) is solved for z^λ as

$$(3.10)\qquad z^\lambda=\frac{2}{1-\alpha^\varepsilon\alpha^{\varepsilon*}}\alpha^\lambda.$$

It follows from (3.9)

$$(3.11)\qquad |\alpha^\lambda\alpha^{\lambda*}|<1,$$

and hence we get a diffeomorphism

$$\tilde{\Psi}: V_\Delta \longrightarrow B^{2m}$$

given by (3.8), where B^{2m} is the domain in E^{2m} satisfying (3.11).

$\tilde{\Psi}$ will be called the *stereographic projection* of CP^m to E^{2m}.

§4. **The induced metric.** We shall calculate the induced metric of the canonical (Kähler) metric

$$ds_\alpha^{\ 2}=2d\alpha^\lambda d\alpha^{\lambda*}$$

of E^{2m} by the stereographic projection

$$\tilde{\Psi}: z^\lambda \longrightarrow \alpha^\lambda=z^\lambda/(f+1).$$

As we have

$$d\alpha^\lambda=\frac{1}{f+1}dz^\lambda-\frac{1}{(f+1)^2}z^\lambda df,$$

the induced metric is given by

$$(4.1)\qquad \tilde{\Psi}^*(ds_\alpha^{\ 2})=\frac{2}{(f+1)^2}\{dz^\lambda dz^{\lambda*}-(df)^2\}.$$

On the other hand, the metric $ds_z^{\ 2}$ of CP^m being (1.3), we have

$$2dz^\lambda dz^{\lambda*}=f^2ds_z^{\ 2}+\frac{2}{f^2}|z^{\lambda*}dz^\lambda|^2.$$

If we substitute the last equation into (4.1) and take account of

$$df=(z^{\lambda*}dz^\lambda+z^\lambda dz^{\lambda*})/2f,$$

$$|z^{\lambda*}dz^\lambda|^2-f^2(df)^2=-(z^{\lambda*}dz^\lambda-z^\lambda dz^{\lambda*})^2/4,$$

then (4.1) reduces to the following

$$(4.2)\qquad \tilde{\Psi}^*(ds_\alpha^{\ 2})=\frac{f^2}{(f+1)^2}ds_z^{\ 2}-\frac{1}{2f^2(f+1)^2}(z^{\lambda*}dz^\lambda-z^\lambda dz^{\lambda*})^2.$$

§5. **The similarity and the inversion in CP^m.** Let ϕ be a similarity at the origin in E^{2m}:

(5.1) $$\phi : \alpha^\lambda \longrightarrow \alpha'^\lambda = c\alpha^\lambda,$$

where c is a positive constant.

A *similarity* $\tilde{\phi}$ of CP^m at $O(\in V_\Delta)$ will be defined by

$$\tilde{\phi} = \tilde{\Psi}^{-1} \circ \phi \circ \tilde{\Psi}.$$

If we take account of (3.9), (3.10), (5.1) and

$$z^\lambda \xrightarrow{\tilde{\Psi}} \alpha^\lambda \xrightarrow{\phi} \alpha'^\lambda \xrightarrow{\tilde{\Psi}^{-1}} z'^\lambda$$

$$\alpha^\lambda = \frac{1}{f+1} z^\lambda, \qquad z'^\lambda = \frac{2}{1-\alpha'^\varepsilon \alpha'^{\varepsilon^*}} \alpha'^\lambda$$

$\tilde{\phi}$ is given in terms of the local coordinate $\{z^\lambda\}$ in V_Δ as follows:

(5.2) $$z'^\lambda = \frac{2c}{(1-c^2)f+1+c^2} z^\lambda.$$

Next we shall induce a transformation of CP^m from an inversion in E^{2m} by $\tilde{\Psi}$. Consider an inversion ϕ in E^{2m} with respect to a hypersphere of origin O_α and radius $r>0$, ϕ is given by an equation of the form

$$\phi : \alpha^\lambda \longrightarrow \alpha'^\lambda = \mathfrak{i}\alpha^\lambda,$$

where $\mathfrak{i}$ is a positive-valued function. As ϕ satisfies

$$(\alpha^\lambda \alpha^{\lambda^*})\ (\alpha'^\mu \alpha'^{\mu^*}) = r^4,$$

we get by virtue of (3.9)

(5.3) $$\mathfrak{i} = \frac{r^2}{\alpha^\varepsilon \alpha^{\varepsilon^*}} = \frac{r^2(f+1)}{f-1}.$$

We shall define $\tilde{\Psi}$ by $\tilde{\Psi} = \tilde{\Psi}^{-1} \circ \tilde{\phi} \circ \Psi$ restricting the value of r to sufficiently small and the domain of ϕ suitably. $\tilde{\phi}$ will be called an *inversion* of CP^m at O. The expression of $\tilde{\phi}$ is found as follows. The equation (5.2) being still true for a non-constant c, we have

$$z'^\lambda = \frac{2\mathfrak{i}}{(1-\mathfrak{i}^2)f+1+\mathfrak{i}^2} z^\lambda.$$

If we substitute (5.3) into the last equation, it follows that

$$z'^\lambda = \frac{2r^2}{f-1-r^4(f+1)} z^\lambda.$$

Putting $c=-1/r^2$, we get for $\tilde{\phi}$

$$z'^\lambda = \frac{2c}{(1-c^2)f+1+c^2} z^\lambda.$$

Comparing this equation with (5.2), we know that the similarity and the inversion of CP^m are given by an equation of the same form:

(5.4) $$z'^{\lambda}=\frac{2c}{(1-c^2)f+1+c^2}z^{\lambda}, \qquad c\neq 0.$$

§6. ***F*-transformation.** Let C^m be the m dimensional complex number space with coordinate $\{z^{\lambda}\}$. The Fubini metric is defined by

(6.1) $$ds_z^{\ 2}=2g_{\lambda\mu^*}dz^{\lambda}dz^{\mu^*}=\frac{1}{S^2}(S\delta_{\lambda\mu}-2kz^{\lambda^*}z^{\mu})dz^{\lambda}dz^{\mu^*}$$
$$=\frac{2}{S}dz^{\lambda}dz^{\lambda^*}-\frac{4k}{S^2}|z^{\lambda^*}dz^{\lambda}|^2,$$

where

$$S=S(u)=1+2ku, \qquad u=z^{\varepsilon}z^{\varepsilon^*},$$

aod k is a non-zero real constant.

Let F^m be the maximal domain of C^m in where S is positive, and we shall call $\{F^m,\ ds_z^{\ 2}\}$ a Fubini space which will be denoted by F^m.

F^m is a Kähler space of constant holomorphic curvature.

Our purpose of this section is to generalize the discussions in §4 and §5 to F^m.

Consider a transformation ϕ of F^m such that

(6.2) $$\phi: z^{\lambda}\longrightarrow w^{\lambda}=t(u)z^{\lambda},$$

where t is a real-valued differentiable function of $u=z^{\varepsilon}z^{\varepsilon^*}$.

It is known [8] that any geodesic through the origin O in F^m is given by

(i) for $k>0$,

$$z^{\lambda}=A^{\lambda}\tan(\sqrt{k}\ s),$$

where A^{λ} are complex numbers satisfying $2kA^{\lambda}\bar{A}^{\lambda}=1$.

(ii) for $k<0$,

$$Z^{\lambda}=A^{\lambda}\tanh(\sqrt{|k|}\ s),$$

where A^{λ} are complex numbers satisfying $2kA^{\lambda}\bar{A}^{\lambda}=-1$.

Thus ϕ leaves invariant each geodesic through O, and hence it is a geodesic transformation at O in the sense of [7].

First we shall get the relation between $ds_z^{\ 2}$ and $\phi^*(ds_w^{\ 2})$. If we put

$$v=w^{\lambda}w^{\lambda^*}=t^2u,$$

it holds that

$$ds_w^{\ 2}=\frac{2}{S(v)}dw^{\lambda}dw^{\lambda^*}-\frac{4k}{S^2(v)}|w^{\lambda^*}dw^{\lambda}|^2.$$

Hence, taking account of

$$dw^\lambda = t'z^\lambda du + t\, dz^\lambda, \qquad t' = dt/du,$$
$$dw^\lambda dw^{\lambda*} = t'(t'u+t)du^2 + t^2 dz^\lambda dz^{\lambda*},$$
$$w^{\lambda*} dw^\lambda = t(t'u\, du + t\, z^{\lambda*} dz^\lambda),$$
$$|w^{\lambda*} dw^\lambda|^2 = t^2\{t'u(t'u+t)\, du^2 + t^2 |z^{\lambda*} dz^\lambda|^2\},$$

we have

$$\phi^*(ds_w{}^2) = \frac{2}{S(v)}\{t'(t'u+t)du^2 + t^2 dz^\lambda dz^{\lambda*}\} - \frac{4kt^2}{S^2(v)}\{t'u(t'u+t)du^2 + t^2|z^{\lambda*}dz^\lambda|^2\}. \tag{6.3}$$

On the other hand, it follows from (6.1) that

$$2dz^\lambda dz^{\lambda*} = S(u)ds_z{}^2 + \frac{4k}{S(u)}|z^{\lambda*}dz^\lambda|^2.$$

Thus, if we substitute the last equation into (6.3), the following equation is obtained:

$$\phi^*(ds_w{}^2) = \frac{t^2 S(u)}{S(v)} ds_z{}^2 + \frac{2}{S(u)S^2(v)} Ⓗ, \tag{6.4}$$

where

$$\begin{aligned} Ⓗ &= (1+2ku(t'(t'u+t)du^2 + 2kt^2(1-t^2)|z^{\lambda*}dz^\lambda|^2 \\ &= (1+2ku)t'(t'u+t)(z^{\lambda*}dz^\lambda - z^\lambda dz^{\lambda*})^2 \\ &\quad + 2\{2(1+2ku)t'(t'u+t) + kt^2(1-t^2)\}\,|z^{\lambda*}dz^\lambda|^2. \end{aligned} \tag{6.5}$$

Now we shall call a transformation ϕ of (6.2) an *F-transformation*, if $t(u)$ satisfies

$$t(u) = \frac{2c}{(1-c^2)f + 1 + c^2},$$

where c is a non-zero real constant and

$$f = \sqrt{S(u)} = \sqrt{1+2ku}, \qquad u = z^\varepsilon z^{\varepsilon*}.$$

This transformation is a generalization of the similarity and the inversion of CP^m.

For an F-transformation ϕ, the coefficient of $|z^{\lambda*}dz^\lambda|^2$ in (6.5) vanishes identically. In fact, it is proved as follows. If we put

$$\rho = \rho(u) = (1-c^2)f + 1 + c^2,$$

then

$$t = \frac{2c}{\rho}$$

and

$$1 - t^2 = 2(1-c^2)\{(1+c^2)f + (1-c^2)(1+ku)\}/\rho^2 \tag{6.6}$$

hold good. As we have

$$f'=k/f, \qquad \rho'=(1-c^2)f'=k(1-c^2)/f,$$

it follows that

$$t'=-2kc(1-c^2)/\rho^2 f,$$

(6.7) $$t'u+t=2c\{(1-c^2)\ (1+ku)+(1+c^2)f\}/\rho^2 f.$$

By (6.6) and (6.7) we can get

$$2(1+2ku)t'(t'u+t)+kt^2(1-t^2)=0,$$

which implies our assertion.

Thus we know that Ⓗ in (6.4) reduces to

$$Ⓗ=(1+2ku)t'(t'u+t)(z^{\lambda^*}dz^\lambda-z^\lambda dz^{\lambda^*})^2$$

for any F-transformation.

§7. The converse problem. Consider a transformation ϕ in $F^m-\{O\}$ given by (6.2), i.e.,

$$\phi: z^\lambda \longrightarrow w^\lambda=t(u)z^\lambda.$$

We assume that ϕ satisfies

$$\phi^*(ds_w{}^2)=\frac{t^2S(u)}{S(v)}ds_z{}^2+\sigma(z^{\lambda^*}dz^\lambda-z^\lambda dz^{\lambda^*})^2$$

identically, where σ is a real-valued function.

The purpose of this section is to prove that the ϕ is an F-transformation.

Under the assumption, the problem is reduced to solving the differential equation for t:

(7.1) $$2(1+2ku)t'(t'u+t)+kt^2(1-t^2)=0$$

by virtue of (6.4) and (6.5).

If we put

$$x=\sqrt{1+2ku}, \qquad y=1/t,$$

then (7.1) becomes the following equation:

(7.2) $$\{(x^2-1)\,p-2xy\}p+y^2-1=0, \qquad p=dy/dx.$$

Differentiating (7.2) with respect to x, we have

$$\{(x^2-1)\,p-xy\}\frac{dp}{dx}=0$$

and hence

(i) $\dfrac{dp}{dx}=0$ or (ii) $\dfrac{p}{y}=\dfrac{x}{x^2-1}$.

Case (i). It follows that $y=Cx+D$, where C and D are constant. Substituting this form of y into (7.2) we get $D^2=C^2+1$ and

$$y=Cx\pm\sqrt{C^2+1}.$$

Therefore if we put

$$c=-C\pm\sqrt{C^2+1},$$

we can get

$$t=\frac{2c}{(1-c^2)x+1+c^2}$$

which shows that ϕ under consideration is an F-transformation.

Case (ii). By integration, we have

$$\log|y|=\frac{1}{2}\log\ |x^2-1|+C. \tag{7.3}$$

If $k>0$, (7.3) gives $y=C\sqrt{x^2-1}$ which and (7.2) lead us to a contradiction $C^2+1=0$. If $k<0$, we have $y=C\sqrt{1-x^2}$, and by (7.2) $C=\pm 1$ follows. Therefore we have

$$t=\pm 1/\sqrt{1-x^2}=\pm 1/\sqrt{-2ku}.$$

Consequently it follows that

$$w^\lambda w^{\lambda*}=t^2 z^\lambda z^{\lambda*}=-1/2k$$

which is contradictory to the diffeomorphism of ϕ.

REMARK. In E^{2m}, let $\phi: z^\lambda \longrightarrow w^\lambda=t(u)z^\lambda$ be a (local or global) diffeomorphism. Then it is easy to see that ϕ satisfies

$$\phi^*(ds_w{}^2)=\rho ds_z{}^2+\sigma(z^{\lambda*}dz^\lambda-z^\lambda dz^{\lambda*})^2$$

if and only if ϕ is a similarity or an inversion with or without composition of the symmetry at O, where ρ and σ are real-valued functions of $\{z^h\}$. In this case, σ actually vanishes.

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