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ON A SPACE OF CONSTANT CURVATURE WITH (f, g, u, v, λ)-STRUCTURE

By U-Hang Ki and Jin Suk Pak

§1. Introduction

In previous paper [3], the present authors proved the theorem;

THEOREM 0.1. *Let M be a complete quasi-normal (f, g, u, v, λ)-structure satisfying one of the following conditions;*

$$\mathscr{L}_U g = 2\alpha\lambda g, \quad du = 2\beta\omega,$$

$$\mathscr{L}_V g = 2\gamma\lambda g, \quad dv = 2\delta\omega,$$

α, β, γ and δ being non-zero functions and $\mathscr{L}_U$ denoting the operator of Lie differentiation with respect to the vector field U. If $\lambda(1-\lambda^2)$ is almost everywhere non-zero function and $\dim M>2$, then M is isometric with an even-dimensional sphere S^{2n}.

The main purpose of the present paper is to prove the following theorem;

THEOREM A. *Let M $(\dim M>2)$ be a complete Riemannian space of constant curvature with (f, g, u, v, λ)-structure. If M satisfies*

$$(0.1) \qquad \nabla_j v_i = \phi f_{ji},$$

ϕ being non-zero differentiable function, and $\lambda(1-\lambda^2)$ is almost everywhere non-zero function, then M is isometric with an even-dimensional sphere.

§2. A space of constant curvature with (f, g, u, v, λ)-structure

Let M be a $2n$-dimensional differentiable manifold of clase C^∞ with an (f, g, u, v, λ)-structure, that is, a tensor field f of type (1,1), a positive definite Riemannian metric g, two 1-forms u_i and v_i (or two vector fields associated with them), and a function λ satisfying

$$(2.1) \qquad f_j^t f_t^h = -\delta_j^h + u_j u^h + v_j v^h,$$

$$(2.2) \qquad f_t^i u^t = -\lambda v^i, \quad u_t f_j^t = \lambda v_j,$$

$$(2.3) \qquad f_t^i v^t = \lambda u^i, \quad v_t f_j^t = -\lambda u_j,$$

(2.4) $u_t u^t = 1-\lambda^2, \quad u_t v^t = 0,$

(2.5) $v_t v^t = 1-\lambda^2, \quad v_t u^t = 0,$

(2.6) $g_{ts} f_j^t f_i^s = g_{ji} - u_j u_i - v_j v_i.$

We put

(2.7) $S_{ji}^{\ \ h} = f_j^t \nabla_t f_i^h - f_i^t \nabla_t f_j^h - (\nabla_j f_i^t - \nabla_i f_j^t) f_t^h + u_{ji} u^h + v_{ji} v^h,$

where

(2.8) $u_{ji} = \nabla_j u_i - \nabla_i u_j, \quad v_{ji} = \nabla_j v_i - \nabla_i v_j,$

∇_j denoting the operator of covariant differentiation with respect to the Riemannian connection. If the tensor $S_{ji}^{\ \ h}$ vanishes, the (f, g, u, v, λ) -structure is said to be normal [1].

Transvecting (2.7) with v_h and using (2.1)~(2.5), we can prove

(2.9) $$S_{ji}^{\ \ h} v_h = v_{ji} - f_j^t f_i^s v_{ts} - \lambda (f_j^t u_{ti} - f_i^t u_{tj})$$
$$- (f_j^t u_i - f_i^t u_j) \nabla_t \lambda + \lambda [(\nabla_j \lambda) v_i - (\nabla_i \lambda) v_j].$$

We now put

(2.10) $f_{jih} = \nabla_j f_{ih} + \nabla_i f_{hj} + \nabla_h f_{ji}.$

If $S_{ji}^{\ \ h} - (f_j^t f_{ti}^{\ \ h} - f_i^t f_{tj}^{\ \ h}) = 0$, then the structure is said to be quasi-normal [2]. Moreover, if a space has the curvature tensor of the form

(2.11) $R_{kjih} = c(g_{kh} g_{ji} - g_{ki} g_{jh}) \qquad (c = \text{constant}),$

then it is said to be a space of constant curvature.

In the sequel we assume $\lambda(1-\lambda^2)$ is almost everywhere non-zero function on M.

§3. Proof of Theorem A

Let M be a Riemannian space of constant curvature with (f, g, u, v, λ)-structure. If M satisfies (0.1), then by transvecting (0.1) with v^i we find

(3.1) $\nabla_j \lambda = \phi u_j.$

Differentiating (3.1) covariantly, we get

$$\nabla_k \nabla_j \lambda = (\nabla_k \phi) u_j + \phi \nabla_k u_j,$$

from which,

(3.2) $0 = (\nabla_k \phi) u_j - (\nabla_j \phi) u_k + \phi (\nabla_k u_j - \nabla_j u_k).$

We have from (0.1)

(3.3) $$\nabla_j v_i - \nabla_i v_j = 2\phi f_{ji},$$

which implies

(3.4) $$0 = (\nabla_k \phi) f_{ji} + (\nabla_j \phi) f_{ik} + (\nabla_i \phi) f_{kj} + \phi f_{kji}.$$

Since $\nabla_k \nabla_j v_i = (\nabla_k \phi) f_{ji} + \phi \nabla_k f_{ji}$, we obtain

$$\nabla_k \nabla_j v_i - \nabla_j \nabla_k v_i = (\nabla_k \phi) f_{ji} - (\nabla_j \phi) f_{ki} + \phi(\nabla_k f_{ji} - \nabla_j f_{ki}),$$

from which, using (2.10), (2.11) and (3.4),

(3.5) $$-c(v_k g_{ji} - v_j g_{ki}) = -(\nabla_i \phi) f_{kj} - \phi \nabla_i f_{kj}.$$

Transvecting (3.5) with f^{kj} and taking account of (2.1) and (2.3), we find

$$-c(\lambda u_i + \lambda u_i) = -(\nabla_i \phi)\ \{2n - 2(1-\lambda^2)\} - \phi \nabla_i \{n - (1-\lambda^2)\},$$

or, using (3.1),

(3.6) $$\{n - (1-\lambda^2)\} \nabla_i \phi = -\lambda(\phi^2 - c) u_i.$$

Using (3.2) and (3.6) and taking account of $n-(1-\lambda^2) \neq 0$, we have

(3.7) $$\phi(\nabla_k u_j - \nabla_j u_k) = 0.$$

Owing to (2.7), (3.3), (3.5) and (3.7), we find

(3.8) $$\phi S_{kji} = -f_k^{\ t}(\nabla_t \phi) f_{ji} + f_j^{\ t}(\nabla_t \phi) f_{ki} + (\nabla_k \phi)(-g_{ji} + u_j u_i + v_j v_i)$$
$$-(\nabla_j \phi)(-g_{ki} + u_k u_i + v_k v_i) + 2(\phi^2 - c) f_{kj} v_i.$$

Transvecting (3.8) with v^i and using (2.2), (2.3), (2.4) and (2.5), we have

(3.9) $$\phi S_{kjt} v^t = \lambda f_k^{\ t}(\nabla_t \phi) u_j - \lambda f_j^{\ t}(\nabla_t \phi) u_k$$
$$-\lambda^2 (\nabla_k \phi) v_j + \lambda^2 (\nabla_j \phi) v_k + 2(\phi^2 - c)(1-\lambda^2) f_{kj}.$$

Multiplying (2.9) by ϕ and substituting (3.1), (3.3) and (3.7) in the equation obtained, we have $\phi S_{kjt} v^t = 0$. Thus, from (3.6) and (3.9), we get

$$0 = -\lambda^3(\phi^2 - c) u_k v_j + \lambda^3 (\phi^2 - c) u_j v_k$$
$$+(1-\lambda^2)\{n - (1-\lambda^2)\}(\phi^2 - c) f_{jk},$$

from which, transvecting $u^j v^k$,

(3.10) $$\phi^2 - c = 0,$$

which means that ϕ is constant. Consequently (3.4) and (3.8) can be respectively written as $f_{kji} = 0$ and $S_{ji}^{\ \ h} = 0$. Hence our structure is quasi-normal. Combining Theorem 0.1, we get the result.

Kyungpook University

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