Kyungpook Math. J.
Volume 12, Number 1
June, 1972

SHIROTA'S THEOREM FOR N-COMPACT SPACES

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We shall consider only Hausdorff spaces.

In [8] Shirota has proved the following theorem: if a completely regular space X admits a complete uniformity and every closed discrete subspace of X is realcompact then X is realcompact. In this paper, we shall give an elementary proof of an analogous result of Shirota's theorem concerning N-compact spaces.

N-compact spaces were introduced by S. Mrowka in [4], where the general concept of an E-compact space was defined: given a space E, a space X is E-compact (respectively E-completely regular) if it is homeomorphic to a closed subspace (respectively, subspace) of E^m for some cardinal number m. For every E-completely regular space X there exists a unique (up to homeomorphism) E-compactification $\beta_E X$ of X such that every continuous function f of X into an E-compact space Y admits a continuous extension f^* from $\beta_E X$ into Y ([6], theorem 4.14). Thus, I-compact spaces (where I is the closed unit interval) are compact Hausdorff spaces; the R-compact spaces are the realcompact spaces or Q-spaces and $\beta_I X$, $\beta_R X$ are respectively the Stone-Čech compactification and the Hewitt-realcompactification of a completely regular space X. For $D=\{0,1\}$ = the two point discrete space; N = the discrete space of all positive integers = the countably infinte discrete space; N^* = the one-point compactification of N. It is easy to derive from the definition that the following conditions are equivalent: a) X is 0-dimensional Hausdorff ("0-dimensional" means "having a base of clopen sets"); b) X is D-completely regular; c) X is N-completely regular; and d) X is N^*-completely regular. It is also clear that every N-compact space is 0-dimensional realcompact and recently, it was shown in [7] that not every 0-dimensional realcompact space is N-compact. But, the realcompactness or N-compactness of a discrete space depends only upon the cardinality of the space, namely [2], [5].

PROPOSITION. *For a discrete space X, the following are equivalent.*

a) X is realcompact

b) X is N-compact

c) The cardinality of X is non-measurable

We recall that a system of subsets of a space is said to be discrete provided that

each point of the space has a neighborhood which intersects at most one member of the system. If $\{F_a : a \in A\}$ is a discrete system of subsets of a space X and $p_a \in F_a$ for each $a \in A$, then $\{p_a : a \in A\}$ is a closed discrete subspace of X. Therefore, according to the above proposition Shirota's theorem can be formulated as follows:

If a completely regular space X admits a complete uniformity and has the property that

(*) the cardinality of every discrete system of subsets of X is non-measurable, then X is realcompact.

The analogous result for Shirota's theorem concerning N-compact spaces reads as follows.

THEOREM. *If an ultranormal space X admits a complete uniformity of clopen coverings and X has the property (*), then X is N-compact.*

(Recall that a space X is ultranormal if for any two disjoint closed sets F, G in X there exists a clopen subset U of X such that $F \subset U$ and $U \cap G = \phi$. The above theorem, in fact, is a consequence of Shirota's theorem and the following one [3]: a strongly 0-dimensional, realcompact space X is N-compact. Note that a completely regular space X is strongly 0-dimensional if for any two disjoint zero sets F, G in X there exists a clopen set U of X such that $F \subset U$ and $U \cap G = \phi$. Clearly an ultranormal space X is strongly 0-dimensional and strongly 0-dimensional space is 0-dimensional. It will be interesting to prove the above theorem for 0-dimensional space instead of ultranormal space.)

We shall give an elementary proof of the above theorem here. We begin by stating the already known results.

LEMMA. *The following are equivalent for a 0-dimensional space X.*

a) X is N-compact

b) ([5], ivb) for each point $p_0 \in \beta_D X \backslash X$ there exists a continuous function $f: \beta_D X \to I$ such that $f(p_0) = 0$ and $f(p) > 0$ for $p \in X$.

c) ([6] corollary 4.19) for each point $p_0 \in \beta_D X \backslash X$ there exists a continuous function f: $\beta_D X \to N^*$ such that $f(p_0) = \infty$ and $f(p) < \infty$ for $p \in X$.

PROOF OF THEOREM. Let X be ultranormal and $\alpha = \{\mathscr{U}\}$ be a complete uniformity of clopen coverings of X. Let $B = \bigcap \{\bigcup \overline{\mathscr{U}} : \mathscr{U} \in \alpha\}$ where $\overline{\mathscr{U}} = \{\overline{U} : U \in \mathscr{U}\}$ and $\overline{U}$ denotes the closure of U in $\beta_D X$. Clearly, $X \subset B$ and we claim that $X = B$. If not,

let $p_0 \in B \backslash X$. Let $\mathscr{F}$ be the collection of all clopen subsets of $\beta_D X$ about p_0. Then $\mathscr{F} \cap X = \{F \cap X : F \in \mathscr{F}\}$ is a center family of clopen subsets of X. For any $\mathscr{U}$ in α, $p_0 \in \bar{U}$ for some $U \in \mathscr{U}$. Thus $\bar{U} \in \mathscr{F}$ and $U = \bar{U} \cap X \in \mathscr{F} \cap X$. Hence $\mathscr{F} \cap X$ is an α-cauchy family and since α is complete, $\cap(\mathscr{F} \cap X) \neq \phi$. But $\cap(\mathscr{F} \cap X) \subset \cap \mathscr{F} = \{p_0\} \subset B \backslash X$. This is a contradiction. Thus $B = X$.

For each point $p_0 \in \beta_D X \backslash X$. Since $X = B$, there exists an $\mathscr{U} \in \alpha$ such that $p_0 \notin \cup \{\bar{U} \mid U \in \mathscr{U}\}$. Since α is a uniformity, there exists a sequence $\{\mathscr{U}_n\}$ in α such that $\mathscr{U}_1$ star refines $\mathscr{U}$ and $\mathscr{U}_{n+1}$ star refines $\mathscr{U}_n$ for $n = 1, 2, \cdots$. Suppose $\mathscr{U} = \{U_a : a \in A\}$ where A is assumed to be a well-ordered set of indices. Then as A. H. Stone showed in the proof of (theorem 1, [9]) there exists a family $F = \{F_{n,a} : n = 1, 2, \cdots ; a \in A\}$ satisfying the following conditions.

a) $\{F_{n,a} : n = 1, 2, \cdots ; a \in A\}$ is a closed covering of X

b) every element of $\mathscr{U}_{n+3}$ does not intersect two elements of $F_n = \{F_{n,a} : a \in A\}$ at the same time, and

c) st $(F_{n,a}, \mathscr{U}_{n+1}) = \cup \{U \in \mathscr{U}_{n+1} ; U \cap F_{n,a} \neq \phi\} \subset U_a$.

Thus, for each $n = 1, 2, \cdots$, $F_n = \{F_{n,a} ; a \in A\}$ is a discrete system of closed subsets of X, so $F = F_1 \cup F_2 \cup F_3 \cup \cdots$ is a σ-discrete closed covering of X and F is a refinement of $\mathscr{U}$. Hence $p_0 \notin \bar{F}_{n,a}$ for each $n = 1, 2, \cdots$, and $a \in A$.

Let $S_n = \cup F_n$ for each $n = 1, 2, \cdots$. We shall distinguish two cases.

Case 1 : $p_0 \notin \bar{S}_n$ for every n. In this case, there exists a continuous function f_n: $\beta_D X \to D$ such that $f_n(p_0) = 0$ and $f_n(p) = 1$ for $p \in S_n$. Setting

$$f(p) = \sum_{n=1}^{\infty} 2^{-n} f_n(p) \qquad \text{for } p \text{ in } \beta_D X$$

We find that f: $\beta_D X \to I$ is continuous and $f(p_0) = 0$, $f(p) > 0$ for $p \in X$. By Lemma, X is N-compact.

Case 2: $p_0 \in \bar{S}_k$ for some k. Since F_k is a discrete system of closed subsets of X, S_k is closed in X and hence each member of F_k is clopen in the subspace S_k of X because $S_k - F_{k,a}$ is closed in X for each $a \in$ A. Consequently, each member of F_k is clopen in the subspace $S_k \cup \{p_0\}$ of $\beta_D X$. Since $p_0 \notin \overline{F_{k,a}}$ for $a \in A$, $\overline{F_{k,a}} \cap (S_k \cup \{p_0\}) = F_{k,a} \cap (S_k \cup \{p_0\}) = F_{k,a}$ so $F_{k,a}$ is closed in $S_k \cup \{p_0\}$ for each $a \in A$. Moreover, since $p_0 \in \bar{S}_k = \overline{S_k \backslash F_{k,a}} \cup \overline{F_{k,a}}$ and $p_0 \notin \overline{F_{k,a}}$ so $p_0 \in \overline{S_k \backslash F_{k,a}}$ for each $a \in A$. Now $(S_k \cup \{p_0\}) \backslash F_{k,a} = (S_k \backslash F_{k,a}) \cup \{p_0\} = [(S_k \cap (\overline{S_k \backslash F_{k,a}})] \cup [\overline{(S_k \backslash F_{k,a})}] \cap \{p_0\}$ $(\overline{S_k \backslash F_{k,a}}) \cap (S_k \cup \{p_0\})$ which is closed in $S_k \cup \{p_0\}$ so $F_{k,a}$ is open in $S_k \cup \{p_0\}$ for

each $a \in A$.

We consider the collection $F_k \cup \{p_0\}$ as a decomposition space of $S_k \cup \{p_0\}$ and let π be the projection of $S_k \cup \{p_0\}$ onto $F_k \cup \{p_0\}$. It follows from the preceding that $\{F_{k,a}\}$ is clopen in $F_k \cup \{p_0\}$ for each $a \in A$. Next, let M be an open set in $F_k \cup \{p_0\}$ and $p_0 \in M$. Then $\pi^{-1}[M] = S_k \cup \{p_0\} \setminus \cup \{F_{k,a} : F_{k,a} \notin M\}$ which is closed in $S_k \cup \{p_0\}$ since each $F_{k,a}$ is open in $S_k \cup \{p_0\}$. Hence M is closed in $F_k \cup \{p_0\}$. This proves that $F_k \cup \{p_0\}$ is 0-dimensional and clearly it is also Hausdorff.

If g: $F_k \to D$ then the function $f = g \circ \pi$ is a continuous function defined on S_k into D. Since S_k is closed in X and X is ultranormal, f admits a continuous function from X into D and in turn, it admits a continuous extension from $\beta_D X$ into D. Hence f admits a continuous extension f^* from $S_k \cup \{p_0\}$ into D. Setting $g^*(p) = g(p)$ for $p \in F_k$ and $g^*(p_0) = f^*(p_0)$, we see that the equality $f^* = g^* \circ \pi$ still holds, therefore g^* is a continuous function from $F_k \cup \{p_0\}$ into D. In other words, every continuous function g from F_k into D admits a continuous extension from $F_k \cup \{p_0\}$ into D and it means that p_0 can be considered as a point from $\beta_D F_k \setminus F_k$.

Since F_k is a discrete system of closed subsets of X, according to the property (*) the cardinality of F_k is non-measurable. Moreover, F_k is a discrete subspace of $F_k \cup \{p_0\}$, by Proposition, F_k is N-compact. Since $p_0 \in \beta_D F_k \setminus F_k$, by Lemma, there exists a continuous function g_0 from $\beta_D F_k$ into N^* such that $g_0(p_0) = \infty$ and $g_0(p) < \infty$ for $p \in F_k$. Define f_0: $S_k \cup \{p_0\} \to I$ as follows:

$$f_0(p) = \frac{1}{(g_0 \circ \pi)(p)} \text{ for } p \in S_k \text{ and } f_0(p_0) = 0$$

We see that f_0 is continuous. By theorem 3.1.(2) of [1], the restriction $f_0 | S_k$ has a continuous extension h from X into $\{0\} \cup \left\{\frac{1}{n} : n \in N\right\}$ such that $h(p) > 0$ for $p \in X$. (Note that if $B = h^{-1}(0) \neq \phi$ then B and S_k are disjoint closed subsets of X, since X is ultranormal, there is a continuous function g: $X \to D$ such that $g | B = 1$ and $g | S_k = 0$. Let $f_1(p) = \max\{h(p), g(p)\}$ for $p \in X$. Then f_1 is a continuous extension of $f_0 | B$ from X into $\{0\} \cup \left\{\frac{1}{n} : n \in N\right\}$ such that $f_0(p) > 0$ for $p \in X$.) Now, since $\{0\} \cup \left\{\frac{1}{n} : n \in N\right\}$ is D-compact, h admits a continuous extension h' from $\beta_D X$ into $\{0\} \cup \left\{\frac{1}{n} : n \in N\right\}$. Clearly, $h'(p) > 0$ for $p \in X$ and $h'(p_0) = f_0(p_0) = 0$ (h' and f_0 agree on S_k and hence they agree on every point of

$\bar{S}_k$). This shows that X is N-compact.

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