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ON THE NON-RIEMANNIAN SPACES

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§1. Introduction.

In an n-dimensional non-Riemannian space A_n the theory of normal tensors based on the system of normal coordinates introduced by O. Veblen [1][1], have been developed by O. Veblen and T.Y. Thomas [2]. In the present paper, considering the space A_n embodied with a field of normal tensors A^i_{jkl}, we shall establish some theorems on the space and shall deal with the affine motion in an A_n. We shall also determine the necessary and sufficient conditions for the equations of the paths admitting homogeneous LFI and QFI, and for the existence of the infinitesimal projective collineation in the space A_n.

The symbolism employed in the present paper is essentially the same as these used in [1], [2] and by Eisenhart [4].

§2. The field of normal tensors A^i_{jkl} and the space A_n.

Considering the symmetric functions Γ^i_{jk} and C^i_{jk} in the space A_n as the connection coefficients in the coordinates x's and its normal coordinates y's respectively, the normal tensors $A^i_{jkl_1\cdots l_r}$ are given by

$$A^i_{jkl_1\cdots l_r}=\left(\frac{\partial^r C^i_{jk}}{\partial y^{l_1}\cdots\partial y^{l_r}}\right)_0, \tag{2.1}$$

where the lower suffix zero indicates the value of the function evaluated at the point P_0 as origin of the normal coördinates y's. From (2.1), it is quite obvious that the components of the normal tensors $A^i_{jkl_1\cdots l_r}$ are symmetric in j and k, and in last r indices. Veblen and Thomas have already shown that any point in the space can be chosen as origin, so the components of these tensors are defined at each point throughout the space and in every coordinate system.

Now for the sake of convenience in our very purpose of present discussion,

1). Numbers in brackets refer to the references at the end of the paper.
2). Eisenhart [4], pp. 68—74.

we take the 4th order mixed normal tensors A^i_{jkl} in consideration and assume that each point of the space A_n is defined by the tensors A^i_{jkl}. So that the space A_n is now embodied with a field of normal tensors A^i_{jkl}. This idea has motivated the author to discuss the various problems in this characteristic space.

The components of the normal tensors A^i_{jkl} in terms of the connection coefficient Γ^i_{jk} are defined by the expression

(2.2) $$A^i_{jkl}=\frac{\partial\Gamma^i_{jk}}{\partial x^l}-\Gamma^i_{jkl}-\Gamma^i_{\beta k}\Gamma^\beta_{jl}-\Gamma^i_{j\beta}\Gamma^\beta_{kl},$$

where

(2.3) $$\Gamma^i_{jkl}\equiv\frac{1}{3}P\Big(\frac{\partial\Gamma^i_{jk}}{\partial x^l}-\Gamma^i_{rk}\Gamma^r_{jl}-\Gamma^i_{jr}\Gamma^r_{kl}\Big)$$
$$=\frac{1}{3}P\Big(\frac{\partial\Gamma^i_{jk}}{\partial x^l}-2\Gamma^i_{jr}\,\Gamma^r_{kl}\Big).$$

In above relation P before an expression indicates the sum of terms obtained by permuting the subscripts cyclically.

We also notice that the tensors A^i_{jkl} satisfy the following identity ([4], p. 70)

(2.4) $$A^i_{jkl}+A^i_{klj}+A^i_{ljk}=0.$$

The Curvature tensor B^i_{jkl} of the space A_n is given by

(2.5) $$B^i_{jkl}=\frac{\partial\Gamma^i_{jl}}{\partial x^k}-\frac{\partial\Gamma^i_{jk}}{\partial x^l}+\Gamma^h_{jl}\,\Gamma^i_{hk}-\Gamma^h_{jk}\,\Gamma^i_{hl}.$$

Since Γ^i_{jkl}, as defined by (2.3) is symmetric in k and l, so it follows from (2.2) and (2.5) that

(2.6) $$B^i_{jkl}=A^i_{jlk}-A^i_{jkl}.$$

Now in accordance with the definition of flat spaces, if the space A_n be flat one, i.e. $B^i_{jkl}=0$, then we have $A^i_{jlk}=A^i_{jkl}$. Applying this fact to the identity (2.4), we get $A^i_{jkl}=0$. Hence we have the

THEOREM 1. *If a non-Riemannian space A_n be a flat space, i.e. $B^i_{jkl}=0$, there does not exist a field of normal tensors A^i_{jkl} in an A_n, and conversely if the space*

A_n embodies a field of normal tensors A^i_{jkl}, the space A_n is necessarily a non-flat space.

Further we remark at this point that a flat space is necessarily a Riemannian space ([4], p. 81), which asserts that the normal coördinates of a flat A_n are now the particular class of Riemannian coördinates in a general Riemannian space. Hence it follows:

THEOREM 2. *There does not exist a field of normal tensors A^i_{jkl} in a general Riemannian spcace V_n.*

§3. A non-Riemannian space of recurrent curvature and the affine motion.

Due to Y. C. Wong [3], an n-dimensional analytic non-Riemannian space A_n of recurrent curvature, for a symmetric affine connection Γ^i_{jk}, has been defined by the characteristic relation

$$B^i_{jkl;m} = K_m B^i_{jkl}, \qquad (K_m \neq 0), \tag{3.1}$$

where a semi-colon followed by an index denotes the covariant differentiation with respect to x's. Such a space A_n is called an AK^*_n-space.

On the other hand, let us consider a field of normal tensors A^i_{jkl}, characterized by a similar type of relation

$$A^i_{jkl;m} = K_m A^i_{jkl}, \tag{3.2}$$

where K_m is a non-zero covariant vector. We shall call such a field, a field of recurrent normal tensors.

Differentiating the relation (2.6) covariantly with respect to x^m, we obtain

$$B^i_{jkl;m} = A^i_{jlk;m} - A^i_{jkl;m}. \tag{3.3}$$

If we introduce the relation (3.2) in (3.3), then, on applying the relation (2.6) in the so obtained result, we can get at once the relation (3.1). Consequently, we have the

THEOREM 3. *When a non-Riemannian space A_n embodies a field of normal tensors A^i_{jkl}, the space A_n is necessarily a space of recurrent curvature if the field possesses the recurrence property.*

Further, we have an useful relation ([4], p. 72)

$$A^i_{jkl;m} = A^i_{jklm} - A^i_{(jkl)m},$$

where $$A^i_{(jkl)m}=(\partial C^i_{jkl}/\partial x^m)_0.$$
In consequence of this, relation (3.3) is changed into the form
$$B^i_{jkl;m}=A^i_{jlkm}-A^i_{jklm}, \qquad \text{(Because } C^i_{jkl}=C^i_{jlk}\text{)}.$$

We now remember that a space A_n with the property that the curvature tensor of A_n are covariant constant, i.e., $B^i_{jkl;m}=0$, is said to be symmetric in the sense of Cartan. In such a case, from above relation, we have the result $A^i_{jlkm}=A^i_{jklm}$, which shows that the tensor A^i_{jlkm} is symmetric in the middle two indices l and k. But, it has already been pointed out that a normal tensor $A^i_{jkl_1\cdots l_r}$ is symmetric in first two indices j and k and in last r indices, so, in the case when A_n is symmetric, the tensor A^i_{jklm} is throughout symmetric in its all lower indices. However, we have the following well known identity ([4], p. 70) satisfied by the tensors A^i_{jklm}
$$A^i_{jklm}+A^i_{jlmk}+A^i_{jmkl}+A^i_{klmj}+A^i_{kmjl}+A^i_{lmjk}=0.$$
In our present case, from the above facts and this identity, we conclude that $A^i_{jklm}=0$. Hence, we can state the

THEOREM 4. *When a non-Riemannian space A_n is a symmetric space, there does not exist a field of normal tensors A^i_{jklm} in an A_n, and conversely, if there does not exist a field of normal tensors A^i_{jklm} in an A_n, the space A_n is necessarily a symmetric space.*

COROLLARY 1. *If an A_n embodies a field of normal tensors A^i_{jklm} the space A_n be a non-symmetric space.*

Next, in the space A_n, let us consider an infinitesimal transformation.

(3.4) $$\bar{x}^i=x^i+\xi^i(x)\delta\tau,$$
where ξ^i is a contravariant vector, and $\delta\tau$ is an infinitesimal constant,
Then we have a deformed space with an affine connection

$\bar{\Gamma}^i_{jk}\overset{\text{def}}{\equiv}\Gamma^i_{jk}+(\pounds\Gamma^i_{jk})\delta\tau$, where $\bar{\Gamma}^i_{jk}$ is the affine connection of the deformed space.

In case of the above transformation, if the original space and the deformed space have the same connection, the transformation (3.4) is called an affine motion of the space A_n. In order that it be the case, it is necessary and sufficient that we have

$$\pounds \Gamma^i_{jk} = 0^{3)}, \tag{3.5}$$

where the notation $\pounds$ denotes the operator of the so called Lie differentiation process.

The Lie derivative of the curvature tensor B^i_{jkl} is given by $\pounds B^i_{jkl} = (\pounds \Gamma^i_{jl})_{;k} - (\pounds \Gamma^i_{jk})_{;l}$, so, due to the condition (3.5), under an affine motion we get necessarily

$$\pounds B^i_{jkl} = 0. \tag{3.6}$$

Now, operating $\pounds$ on both the sides of (2.6) and applying the condition (3.6) in the result, we obtain

$$\pounds A^i_{jlk} = \pounds A^i_{jkl}. \tag{3.7}$$

Thus, it concludes the

THEOREM 5. *When a non-Riemannian space A_n embodies a field of normal tensors A^i_{jkl}, in order that the transformation* (3.4) *be an affine motion of A_n, it is necessary and sufficient that we have*

$$\pounds A^i_{jlk} = \pounds A^i_{jkl}.$$

Further we see that for any general tensor T^i_{jkh}, the following relation[4] holds good

$$(\pounds T^i_{jkh;l} - (\pounds T^i_{jkh})_{;l} = (\pounds \Gamma^i_{al}) T^a_{jkh} - (\pounds \Gamma^a_{jl}) T^i_{akh} - (\pounds \Gamma^a_{kl}) T^i_{jah} - (\pounds \Gamma^a_{hl}) T^i_{jka}. \tag{3.8}$$

In the present case, employing this formula for the curvature tensor B^i_{jkl}, we get from (3.5) and (3.6)

$$\pounds (B^i_{jkl;m}) = 0. \tag{3.9}$$

Introducing the relation (3.3) in (3.9) and using (3.2) and (3.7), we obtain after some simple calculation

$$(A^i_{jlk} - A^i_{jkl}) \pounds K_m = 0.$$

Since our space is not flat one, i.e. $B^i_{jkl} = A^i_{jlk} - A^i_{jkl} \neq 0$, therefore $\pounds K_m = 0$.

Now, if we again use the relation (3.8) for the tensor A^i_{jkl} and apply the relat-

3). See Yano [5], relation (2.21), page 8.

4). Yano [5], page 16.

ion (3.2) and the condition $\pounds K_m=0$ in the result, then, under an affine motion, we get a requisite condition $(\pounds A^i_{jkl})_{;m}=K_m(\pounds A^i_{jkl})$. Hence, we have the

THEOREM 6. *When an AK^*_n-space embodies a field of recurrent normal tensors A^i_{jkl}, in order that the field may admit an affine motion of the form* (3.4), *it is necessary and sufficient that* $(\pounds A^i_{jkl})_{;m}=K_m(\pounds A^i_{jkl})$.

§4. The equations of the paths admitting independent homogeneous linear first integrals and quadratic first integrals.

In an n-dimensional space A_n, let us consider the equations of the paths

$$\frac{d^2x^i}{ds^2}+\Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}=0. \tag{4.1}$$

If each integral of the equations of the above paths satisfies the condition

$$a_{r_1r_2\cdots r_m}\frac{dx^{r_1}}{ds}\frac{dx^{r_2}}{ds}\cdots\cdots\frac{dx^{r_m}}{ds}=C\,;\quad (C\text{: a constant}),$$

the equations (4.1) are said to admit a homogeneous first integral of the mth degree. But for the sake of brevity, here on one hand we shall consider the integral of the first degree, that is an independent homogeneous linear first integral (LFI)

$$a_i\frac{dx^i}{ds}=C_1\,;\quad (C_1\text{: a constant}). \tag{4.2}$$

If the equations of the paths (4.1) admit a homogeneous LFI (4.2), it is necessary that

$$a_{i;j}+a_{j;i}=0. \tag{4.3}$$

Differentiating this relation covariantly with respect to x^k, we have

$$a_{i;jk}+a_{j;ik}=0. \tag{4.4}$$

Now, we consider that $T^{i_1\cdots i_r}_{j_1\cdots j_s}$ and $t^{i_1\cdots i_r}_{j_1\cdots j_s}$ are the components of a tensor in the coordinate system x^i and its normal coordinates y^i respectively, then the general relation ([4], p. 74) connecting covariant derivatives of the tensor $T^{i_1\cdots i_r}_{j_1\cdots j_s}$ and its extensions of the second order can be given as follows;

$$T^{i_1\cdots i_r}_{j_1\cdots j_s;kl}=T^{i_1\cdots i_r}_{j_1\cdots j_s:kl}+\sum_p^{1,\cdots,r}T^{i_1\cdots i_{p-1}hi_{p+1}\cdots i_r}_{j_1\cdots j_s}A^{ip}_{hkl}-\sum_q^{1,\cdots,s}T^{i_1\cdots i_r}_{j_1\cdots j_{q-1}hj_{q+1}\cdots j_s}A^h_{j_qkl}, \tag{4.5}$$

where $T^{i_1\cdots i_r}_{j_1\cdots j_s:kl}$ are the second order extensions of the tensor $T^{i_1\cdots i_r}_{j_1\cdots j_s}$ defined by

$$T^{\,i_1\cdots i_r}_{\,j_1\cdots j_s kl}=\left(\frac{\partial^2 t^{i_1\cdots i_r}_{j_1\cdots j_s}}{\partial y^k \partial y^l}\right)_0. \tag{4.6}$$

The colon followed by the indices denotes the extension process of the tensors of the same order as the sum of the given number of indices following a colon, and the lower suffix zero indicates the value of the function evaluated at the point P_0 as origin. Since any point can be chosen as P_0, so the relation (4.5) holds good for any point throughout the space.

Similarly if we assume that a_i and α_i are the components of a tensor of order one in x^i coordinates and its normal coordinates y^i respectively, then, with the help of (4.5) and (4.6) the covariant derivatives of a_i in terms of the second order extension can be given by

$$a_{i;jk}=a_{i:jk}-a_h A^h_{ijk}, \tag{4.7}$$

where
$$a_{i:jk}=\left(\frac{\partial^2 \alpha^i}{\partial y^j \partial y^k}\right)_0.$$

Now, adding the relation (4.4) with the first of the following.

$$a_{k;ij}+a_{i;kj}=0, \qquad a_{j;ki}+a_{k;ji}=0$$

and subtracting the other, then, on making the use of the relation (4.7) and the identity (2.4), we shall get at once

$$a_{i:jk}=-a_h(A^h_{jki}+A^h_{kji}) \tag{4.8}$$

Making the cyclic change in the indices i, j, k, of the relation (4.8) and adding the so obtained two relations with that of (4.8), we get at last

$$a_{i:jk}+a_{j:ki}+a_{k:ij}=0, \tag{4.9}$$

where we have used the identity (2.4) again. Hence we have the

THEOREM 7. *When a non-Riemannian space A_n embodies a field of normal tensors A^i_{jkl}, in order that the equations of the paths* (4.1) *of the space A_n admit homogeneous LFI* (4.2), *it is necessary and sufficient that the identity* (4.9) *necessarily holds good.*

Next, we consider the integral of the second degree, that is an independent homogeneous quadratic first integral (QFI)

$$a_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=C_2: \quad (C_2\text{: a constant}). \tag{4.10}$$

If the equations of the paths (4.1) admit a homogeneous QFI (4.10), it is necessary that

(4.11) $$a_{ij;k}+a_{jk;i}+a_{ki;j}=0.$$

Now, for the sufficiency that the equations of the paths (4.1) admit a homogeneous QFI (4.10), we determine a concrete condition. For this purpose, first, differentiating (4.11) covariantly, we obtain

(4.12) $$a_{ij;kl}+a_{jk;il}+a_{ki;jl}=0.$$

Next, by making the use of the formula (4.5), one may easily have the

(4.13) $$a_{ij;kl}=a_{ij:kl}-a_{hj}A^h_{ikl}-a_{ih}A^h_{jkl}.$$

Similarly, for the other terms in (4.12), consecutive expressions are obtained as

(4.14) $$a_{jk;il}=a_{jk:il}-a_{hk}A^h_{jil}-a_{jh}A^h_{kil},$$

(4.15) $$a_{ki;jl}=a_{ki:jl}-a_{hi}A^h_{kjl}-a_{kh}A^h_{ijl}.$$

Here we notice that the second extension $a_{ij:kl}$ of the tensor a_{ij} is symmetic in i and j, and in k and l. Also the normal tensor A^h_{jkl} is symmetric in its first two lower indices. So adding (4.13), (4.14), (4.15) and using (4.12), we can get with ease

(4.16) $$a_{ij:kl}+a_{jk:il}+a_{ki:jl}=2(a_{hj}A^h_{ikl}+a_{hk}A^h_{jil}+a_{hi}A^h_{kjl}).$$

Making the cyclic change of the indices i, j, k and l in (4.16), three other expressions will be obtained successively. Adding these three with that of (4.16) and using the identity (2.4), and on arranging the terms, we get finally

(4.17) $$(a_{ij:kl}+a_{kl:ij})+(a_{jk:li}+a_{li:jk})+(a_{ki:lj}+a_{lj:ki})=0.$$

From which, obviously we see that, if $a_{ij:kl}=-a_{kl:ij}$, the relation (4.17) is identically satisfied. Hence we have the

THEOREM 8. *When a non-Riemannian space A_n embodies a field of normal tensors A^i_{jkl}, in order that the equations of the paths* (4.1) *of the space A_n admit homogeneous QFI* (4.10), *it is necessary and sufficient that $a_{ij:kl}=-a_{kl:ij}$, that is to say, the second extension $a_{ij:kl}$ of the tensor a_{ij} be antisymmetric in its pair of indices ij and kl.*

§5. Conditions for the existence of infinitesimal collineations in an affinely connected space.

If some point transformations transform the points of an affinely connected manifold into points of the manifold such that the paths are transformed into paths. The transformations are called collineations.

Let us consider an infinitesimal point transformation

$$\bar{x}^i = x^i + \xi^i(x)\delta\tau, \tag{5.1}$$

where ξ^i are the function of x's and $\delta\tau$ is an infinitesimal. If (5.1) be the collineation in an A_n, the necessary conditions for this are

$$\xi^h_{;ij} = \xi^k B^h_{ijk} + \delta^h_j \varphi_i + \delta^h_i \varphi_j, \tag{5.2}$$

and

$$\xi^h_{;ij} = \xi^k \; B^h_{ijk}. \tag{5.3}$$

When a set of functions ξ^i are a solution of (5.2), where $\varphi_i \neq 0$, the collineation preserves the projective properties and is called an infinitesimal projective collineation. On the other hand, when the set of functions ξ^i are a solution of (5.3) and the collineation (5.1) preserves the affine properties of the space, the collineation (5.1) is called an infinitesimal affine collineation in the space A_n.

The second order covariant derivative of a vector ξ^i and its second extension may be connected by the relation

$$\xi^h_{;ij} = \xi^h_{:ij} + \xi^k A^h_{kij}. \tag{5.4}$$

Introducing this relation on the left of (5.2) and taking help of (2.6), the relation (5.2) may be rewritten in the form

$$\xi^h_{:ij} + \xi^k A^h_{kij} = \xi^k (A^h_{ikj} - A^h_{ijk}) + \delta^h_j \varphi_i + \delta^h_i \varphi_j. \tag{5.6}$$

Interchanging the indices i and j in this relations, and adding the resulting equation with (5.6), we get

$$\xi^h_{;ij} + \xi^h_{;ji} + \xi^k (A^h_{kij} + A^h_{kji}) = \xi^k (A^h_{ikj} - A^h_{ijk} + A^h_{jki} - A^h_{jik}) + 2(\delta^h_j \varphi_i + \delta^h_i \varphi_j).$$

or

$$\xi^h_{:ij} + \xi^h_{:ji} = \xi^k (A^h_{ikj} - A^h_{ijk} + A^h_{jki} - A^h_{jik} - A^h_{kij} - A^h_{kji}) + 2(\delta^h_j \varphi_i + \delta^h_i \varphi_j). \tag{5.7}$$

Since the tensors A^h_{ijk} are symmetric in the first two lower indices i and j, so because of the condition $\xi^h_{:ij} = \xi^h_{:ji}$, from (5.7) we shall obtain

$$\xi^h_{:ij} = -\xi^k A^h_{ijk} + \delta^h_j \varphi_i + \varphi_i \varphi_j. \tag{5.8}$$

Contracting the relation (5.8) with respect to the indices h and i, we find that

$$0 = -\xi^k A^h_{hjk} + (n+1)\varphi_j$$

From which we conclude that

$$\varphi_j = \frac{1}{n+1} \xi^k A^h_{jhk}, \qquad \text{(Because } A^h_{hjk} = A^h_{jhk}\text{)}. \tag{5.9}$$

Hence, we can summarize the above result as follows:

THEOREM 9. *A non-Riemannian space A_n embodied with a field of normal tensors A^h_{ijk}, admits an infinitesimal projective collineation if and only if the covariant vector φ_j satisfies the relation*

$$\varphi_j=\frac{1}{n+1}\xi^k A^h_{jhk}.$$

Next, we remark that, when φ_j is zero, the projective collineation becomes an affine collineation. In such a case, we conclude that $\xi^k A^h_{jhk}=0$. Consequently, it follows

COROLLARY 2. *When a non-Riemannian space A_n embodies a field of normal tensors A^h_{ijk}, in order that an infinitesimal projective collineation admitted in A_n becomes an infinitesimal affine collineation in the space, it is necessary and sufficient that we assume $\xi^k A^h_{jhk}=0$.*

Furthermore, we shall determine a concrete result for the existence of the infinitesimal collineation in the space A_n. For this very purpose, first we have to assume $\xi^h_{;i}=\eta^h_i$.

Introducing this concept in (5.2) and using the relations (2.6) and (5.9), we find that

$$\eta^h_{i;j}=\xi^l\,(A^h_{ilj}-A^h_{ijl})+\frac{1}{n+1}\,[\delta^h_j\xi^l A^r_{irl}+\delta^h_i\xi^l A^r_{jrl}].$$

Differentiating this relation covariantly with respect to x^k, then with the help of relation $\xi^h_{;i}=\eta^h_i$, we obtain

$$\begin{aligned}\eta^h_{i;jk}=\eta^l_k(A^h_{ilj}-A^h_{ijl})+\xi^l(A^h_{ilj;k}-A^h_{ijl:k})+\frac{1}{n+1}\,[\delta^h_j\,(\eta^l_k A^r_{irl}+\xi^l A^r_{irl;k})\\ +\delta^h_i\,(\eta^l_k A^r_{jrl}+\xi^l A^r_{jrl;k})].\end{aligned}\tag{5.10}$$

Also, we have the following relation connecting the second order covariant derivative $\eta^h_{i;jk}$ of a tensor η^h_i and its second extension $\eta^h_{i:jk}$,

$$\eta^h_{i;jk}=\eta^h_{i:jk}+\eta^l_i\;A^h_{ljk}-\eta^h_l\,A^l_{ijk}.\tag{5.11}$$

Introducing the relation (5.11) on the left of (5.10), we have

$$\begin{aligned}\eta^h_{i:jk}+\eta^l_i\,A^h_{ljk}-\eta^h_l\,A^l_{ijk}=\eta^l_k(A^h_{ilj}-A^h_{ijl})+\xi^l(A^h_{ilj;k}-A^h_{ijl;k})\\ +\frac{1}{n+1}\,[\delta^h_j\,(\eta^l_k A^r_{irl}+\xi^l A^r_{irl:k})+\delta^h_i\,(\eta^l_k A^r_{jrl}+\xi^l A^r_{jrl;k})].\end{aligned}$$

Now, if we interchange the indices j and k in this relation, then on subtracting the later equation with the former one, we get at last

$$\begin{aligned}\eta^l_i A^h_{ljk}-\eta^h_l A^l_{ijk}-\eta^l_i A^h_{lkj}+\eta^h_l A^l_{ikj}&=\eta^l_k(A^h_{ilj}-A^h_{ijl})\\&-\eta^l_j(A^h_{ilk}-A^h_{ikl})+\xi^l(A^h_{ilj;k}-A^h_{ijl;k})-\xi^l(A^h_{ilk;j}-A^h_{ikl;j})\\&+\frac{1}{n+1}[\delta^h_j(\eta^l_k A^r_{irl}+\xi^l A^r_{irl;k})+\delta^h_i(\eta^l_k A^r_{jrl}+\xi^l A^r_{jrl;k})]\\&-\frac{1}{n+1}[\delta^h_k(\eta^l_j A^r_{irl}+\xi^l A^r_{irl;j})+\delta^h_i(\eta^l_j A^r_{krl}+\xi^l A^r_{krl;j})].\end{aligned}$$

where we have used $\eta^h_{i:jk}=\eta^h_{i:kj}$. The contraction of $h=i$ in this equation gives us

$$(5.13)\qquad\begin{cases}\eta^l_r A^r_{ljk}-\eta^r_l A^l_{rjk}-\eta^l_r A^r_{lkj}+\eta^l_r A^l_{rkj}\\ \qquad=\eta^l_k A^r_{rlj}+\xi^l A^r_{rlj;k}-\eta^l_j A^r_{rlk}-\xi^l A^r_{rlk;j}.\end{cases}$$

Next, we put $\eta^l_j=\xi^l_{;j}$ and use the notation $\eta^l_r A^r_{ljk}\equiv\theta_{jk}$ in (5.13), we find that

$$\xi^l_{;k}A^r_{rlj}+\xi^l A^r_{rlj;k}=\xi^l_{;j}A^r_{rlk}+\xi^l A^r_{rlk;j},$$

which yields the result

$$(\xi^l A^r_{rlj})_{;k}=(\xi^l A^r_{rlk})_{;j}.$$

Hence, we have the

THEOREM 10. *When a non-Riemannian space A_n embodies a field of normal tensors A^h_{ijk}, in order that the space A_n admits an infinitesimal projective collineation, it is necessary and sufficient that we have*

$$(\xi^l A^r_{rlj})_{;k}=(\xi^l A^r_{rlk})_{;j}.$$

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