A Note on Equivalence Relations and Continuous Functions

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1. Introduction

The theory of set, which was founded by **G. Cantor**(1845—1918), is not only one of the most important tools in modern mathematics, but also the greatest creations of the human mind. It has recently become an essential part of mathematical back ground of high school students and teachers.

In this note, my aim is to provide some fundamental properties of sets, relations and functions. The first chapter presents required concepts of set theory. The second chapter treats relations, equivalence relations and quotient sets. The third chapter, on topological spaec, defines continuous function and on **connected** space proves the generalization of intermediate values theorem with properties of continuous function.

·Prerequisites for reading this note are only the concepte of "least upper bound" in analysis, and the validity of the general distributive law of $\bigcup$ over $\bigcap$ in proof of 4.12 theorem can be found in Dugundji [2] 9.6, p.55.

2. Sets

The theory of sets has been described axiomatically in terms of the notion "member of set".

Because to build complete theory of sets from these axiomis a long, difficult process, we take the notion of set as intuitively. We think of a set as something made up by all the objects that satisfy some given condition, such as the set of prime numbers, the set of points on a line, or the set of objects named in a given list. The object making up the set are called the element of the set and may themselves be sets, as in the set of all lines in the plane.

Roughly speaking, a set (class, family) is any identificable collection of objects of any sort. Sets will be denoted by capital Roman letters: **A,B,C,X,Y,Z,**. Elements (points, members) of sets will be denoted by small letters: $a,b,c,x,y,z.$ A set is often defined by some property of its elements. We will write $\{x|p(x)\}$ [where $p(x)$ is some proposition about x] to denote the set of all x such that $p(x)$ is true. For example

$A=\{a, b, c, d\}$

$B=\{x|x \text{ is an integer}, x>0\}$

$=\{1, 2, 3, \cdots\cdots\}$

$C=$ "open interval from a to b"

$=\{x|a<x<b\}=(a, b)$

$X=\{x|x^2-2x+1=0\}=\{1\}$

If the object x is an element of the set **A**, we will write $x \in \mathbf{A}$; while $x \notin \mathbf{A}$ will mean that x is not in **A**.

2, 1. [Def] Let **A** and **B** be sets such that for all x, $x \in \mathbf{A}$ implies $x \in \mathbf{B}$. Then **A** is called a **subset** of **B** and we write $\mathbf{A} \subset \mathbf{B}$. If $\mathbf{A} \subset \mathbf{B}$ and $\mathbf{B} \subset \mathbf{A}$, then we write $\mathbf{A} = \mathbf{B}$. If $\mathbf{A} \subset \mathbf{B}$ and $\mathbf{A} \neq \mathbf{B}$, we say that **A** is **proper subset** of **B** and we write $\mathbf{A} \subsetneq \mathbf{B}$.

We write ϕ for the empty set: it has no elements at all. Empty set is a subset of every set, and $\mathbf{A} \subset \mathbf{A}$ for every set **A**, the fact that the empty set is a subset of every set is based on a point of logic: by 2. 1, it is clear that if **A** is not a subset of **B**, the following statement must be true; "There is an element x such that $x \in \mathbf{A}$ and $x \notin \mathbf{B}$." But if **A** is empty, there is no x such that $x \in \mathbf{A}$, and the above statement is false (See 2, 9, exercise)

2, 2. [Def] If **A** and **B** are sets, then we define $\mathbf{A} \cup \mathbf{B}$ as the set $\{x \mid x \in \mathbf{A} \text{ or } x \in \mathbf{B}\}$ (here "or" is used the sense of "and/or"), and we call the **Union** (or **Cup**) of **A** and **B**. Let $\mathfrak{A}$ be a family of sets, then we define $\bigcap \mathfrak{A} = \{x \mid x \in \mathbf{A} \text{ for some } \mathbf{A} \in \mathfrak{A}\}$. We define $\mathbf{A} \cap \mathbf{B}$ as the set $\{x \mid x \in \mathbf{A} \text{ and } x \in \mathbf{B}\}$, and we call $\mathbf{A} \cap \mathbf{B}$ the **intersection** (or **Cap**) of **A** and **B**, also we define $\bigcap \mathfrak{A} = \{x \mid x \in \mathbf{A} \text{ for all } \mathbf{A} \in \mathfrak{A}\}$.

If $\mathbf{A} \cap \mathbf{B} = \phi$, that is. if **A** and **B** do not have any element in common, then **A** and **B** are said to **disjoint.**

For example

1) Let $\mathbf{A} = \{1, 2, 3\}$, $\mathbf{B} = \{1, 2, \{1, 2\},\}$ then

$\mathbf{A} \cup \mathbf{B} = \{1, 2, 3 \{1, 2\}, \{1, 2, 3\}\}$ $\mathbf{A} \cap \mathbf{B} = \{1, 2\}$

2) Let $\mathbf{N} = \{x \mid x \text{ is a natural number}\}$

$\mathbf{A}_n = \{x \mid x \text{ is a real number}, \ |x| < \frac{1}{n}, \ n \in \mathbf{N}\}$,

$\mathfrak{A} = \{\mathrm{A}_n \mid n \in \mathrm{N}\}$,

$\mathbf{B}_n = \left\{x \mid -\frac{1}{n} \leqq x \leqq \frac{1}{n}\right\}$,

$\mathbf{B} = \{\mathrm{B}_n \mid n \in \mathrm{N}\}$

then $\bigcap \mathfrak{A} = \bigcap_{n=1}^{\infty} \mathrm{A}_n = \{0\}$

$\bigcup \mathbf{B} = \bigcup_{n=1}^{\infty} \mathrm{B}_n = \{x \mid -1 \leqq x \leqq 1\} = [-1, 1]$

2, 3. [Def] Unfortnnately set theory leads to contradiction when one uses sets that are "too big," such as to speak of a set which contains everything, as Frege and Russell supposed. Therefore, in most of our discussions, all sets are subsets of a some fixed set; we call this set the **Universal set** and denoted by **X,Y.**

2, 4. [Def] Let A and B are sets, we define **A—B as the set** $\{x \mid x \in \mathrm{A} \text{ and } x \notin \mathrm{B}\}$ and we call A—B the **difference** of A and B, and we define **A′** as the difference of the universal set X and A, that is, $\mathbf{A}' = \{x \mid x \in \mathrm{X} \text{ and } x \notin \mathrm{A}\}$. **A′** is called **complement** of A. For example, Let $\mathrm{X} = \{1, 2, 3, 4, 5, 6\}$, $\mathrm{A} = \{1, 2, 3, 4\}$, $\mathrm{B} = \{3, 4, 5\}$ then $\mathrm{A} - \mathrm{B} = \{1, 2\}$, $\mathrm{B} - \mathrm{A} = \{5\}$, $\mathbf{A}' = \{5, 6\}$ and $\mathbf{B}' = \{1, 2, 6\}$

2, 5. [Def] For a set A, the family of all subsets of **A** is a well defined family of sets which is known as the **power set** of A and is denoted by $\mathfrak{P}(\mathbf{A})$. For example, if $\mathrm{A} = \{1, 2\}$, then $\mathfrak{P}(\mathbf{A}) = \{\phi, \mathrm{A}, \{1\}, \{2\}.\}$

Let $\alpha(\mathrm{A})$ denote the number of elements of A, then $\alpha(\mathrm{A})$ is a nonnegative integer, and the foll-

...ow: propositions are true.

1) $\alpha(A\cap B)+\alpha(A\cup B)=\alpha(A)+\alpha(B)$

2) If $\alpha(A)=n$, then $\alpha(\mathfrak{P}(A))=2^n$

Proof. Since ${}_nC_m$ is the number of elements of family $\mathfrak{P}(A)$ which contains m elements of the set A,

$$\alpha(\mathfrak{P}(A))={}_nC_0+{}_nC_1+\cdots\cdots+{}_nC_m+\cdots\cdots+{}_nC_n=(1+1)^n=2^n$$

For exampl, Let $A=\{1,2,a\}$, then $\mathfrak{P}(A)=\{\phi, \{1\}, \{2\}, \{a\}, \{1,2\}, \{2,a\}, \{a,1\}, A\}$. just as $\alpha(\mathfrak{P}(A))=2^3=8$

2, 6. Theorem 1. *Laws of the algebra of sets*

Let A, B, C be any subsets of X, then we have:

(1) $A\cup A=A$	(1′) $A\cap A=A$	Idempotent Laws
(2) $A\cup B=B\cup A$	(2′) $A\cap B=B\cap A$	Commutative 〃
(3) $(A\cup B)\cup C=A\cup(B\cup C)$	(3′) $(A\cap B)\cap C=A\cap(B\cap C)$	Associative 〃
(4) $A\cup(B\cap C)=(A\cup B)\cap(A\cup C)$	(4′) $A\cap(B\cup C)=(A\cap B)\cup(A\cap C)$	Distributive 〃
(5) $A\cup\phi=A$	(5′) $A\cap X=A$	Identity 〃
(6) $A\cup X=X$	(6′) $A\cap\phi=\phi$	〃 〃
(7) $A\cup A'=X$	(7′) $A\cap A'=\phi$	Complement 〃
(8) $(A')'=A$	(8′) $X'=\phi$, $\phi'=X$	〃 〃
(9) $(A\cup B)'=A'\cap B'$	(9′) $(A\cap B)'=A'\cup B'$	De Morgan's 〃

Proof. 4′) $A\cap(B\cup C)=\{x\mid x\in A \text{ and } x\in B\cup C\}$

$=\{x\mid x\in A \text{ and } (x\in B \text{ or } x\in C)\}$

$=\{x\mid (x\in A \text{ and } x\in B) \text{ or } (x\in A \text{ and } x\in C)\}$

$=\{x\mid x\in A\cap B \text{ or } x\in A\cap C\}=(A\cap B)\cup(A\cap C)$

8) $x\in(A')'\Longleftrightarrow x\notin A'\Longleftrightarrow x\in A$

9) $x\in(A\cup B)'\Longleftrightarrow x\notin A'(\cup B')\Longleftrightarrow x\notin A$ and $x\notin B\Longleftrightarrow x\in A'$ and $x\subset B'\Longleftrightarrow x\in A'\cap B'$

9′) By (8)(9), $(A'\cup B')'=[(A')'\cap(B')']=A\cap B$

Hence by(8) $A'\cup B'=(A\cap B)'$

Another Proof is very simple.

2, 7. theorem 2

Each of the following conditions is equivalent to $A\subset B$

(1) $A\cap B=A$	(2) $B'\subset A'$	(3) $B\cup A'=X$
(4) $A\cup B=B$	(5) $A\cap B'=\phi$	

Proof: (1) $\left.\begin{array}{l}A\subset B\Longleftrightarrow(x\in A\to x\in B)\Longleftrightarrow(x\in A\to x\in A\cap B)\Longleftrightarrow A\subset A\cap B\\ x\in A\cup B\to x\in A\Longleftrightarrow A\cup B\subset A\end{array}\right\}\Longleftrightarrow A\cap B=A$

(2) $A\subset B\Longleftrightarrow(x\in A\to x\in B)\Longleftrightarrow(x\notin B\to x\notin A)\Longleftrightarrow(x\leftarrow B'\to x\leftarrow A')\Longleftrightarrow B'\subset A'$

(3) Since every set is a subset of the universal set X, $B\cup A'\subset X$, thus only the converse inclusion requires. If $x\in X$ then x is contained in either $A(A\subset B)$ or A'. In either case, $x\in B\cap A'$. It follows $X\subset B\cap A'$.

(4) $\left.\begin{array}{l}B\subset A\cup B \text{ is trivial by 2, 2. [Def]}\\ A\subset B\Longleftrightarrow(x\in A\cup B\to x\in B)\Longleftrightarrow A\cup B\subset B\end{array}\right\}\Longleftrightarrow A\cup B=A$

(5) $B \subsetneqq A \Longleftrightarrow (\exists x : x \in A$ and $x \in B')$ (there exist x such that $x \in A$ and $x \in A$ and $x \in B'$)
$\Longleftrightarrow (\exists x : x \in A \cap B') \Longleftrightarrow A \cap B' \neq \phi$

2, 8. exercise

(1) $B - A = B \cap A'$

(2) $(A - B) \cap B = \phi$

(3) $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Proof: (1) $B - A = \{x \mid x \in B,\ x \notin A\} = \{x \mid x \in B,\ x \in A'\} = B \cap A'$

(2) $(A - B) \cap B = \{x \mid x \in B,\ x \in A - B\} = \{x \mid x \in B$ and $(x \in A$ and $x \notin B)\} = \phi$

(3) $(A - B) \cup (B - A) = (A \cap B') \cup (B \cap A') = \{A \cup B') \cup B\} \cap \{(A \cap B') \cup A'\}$
$= (A \cup B) \cap (B' \cup B) \cap (A \cup A') \cap (B' \cup A') = (A \cup B) \cap X \cap X \cap (A \cap B)'$
$= (A \cup B) \cap (A \cap B)' = (A \cap B) - (A \cap B)$

2, 9. exercise

Empty set ϕ is a subset of every set.

Proof Let A is any subset of universal set X, then $A' \subset X$. Hence

$$A' \subset X \underset{2.7(2)}{\Longleftrightarrow} (A')' \supset X' \underset{2.6(8)}{\Longleftrightarrow} A \supset \phi$$

Another proof

$\phi \subset A \Longleftrightarrow (x \in \phi \Longleftrightarrow x \in A)$
$\Longleftrightarrow (x \notin A \Rightarrow x \notin \phi)$: Since ϕ has no element at all, the latter proposition is true.

3. Relations and equivalence relations

3, 1. [Def] Let **X** and **Y** be sets. The **Cartesian Product** of **X** and **Y** is the set $\mathbf{X} \times \mathbf{Y}$ of all ordered pairs (x, y) such that $x \in X$ and $y \in Y$.

We write $(x, y) = (u, v)$ if and only if $x = u$ and $y = v$.

Thus $(1, 2) \neq (2, 1)$ while $\{1, 2\} = \{2, 1\}$.

Clearly, Euclidian coordinate is the Cartesian product of the set of real numbers X and Y.

3, 2. [Def] A relation is any set af ordered pairs. Thus a relation is any set which is a subset of the **Cartesian product** of two sets. Since ϕ is a subset of every set, ϕ is a relation.

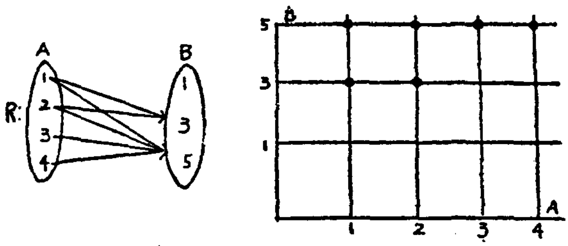
3, 3. [Def] Let **R** be any relation. We define $(x, y) \in \mathbf{R} \subset X \times Y$ if and only if "x is related to y" written $x\mathbf{R}y$, and define the **Domain** of **R** to be the set Dom $\mathbf{R} = \{x \mid (x, y) \in \mathbf{R}\}$, the **Range** of **R** to be the set Ran $\mathbf{R} = \{y \mid (x, y) \in \mathbf{R}\}$. The symbol $\mathbf{R}^{-1}$ denotes the **inverse** of $R : R^{-1} = \{(y, x) \mid (x, y) \in R\}$.

3, 4. exercise

Proof $A \times (B \cap C) = (A \times B) \cap (A \times C)$

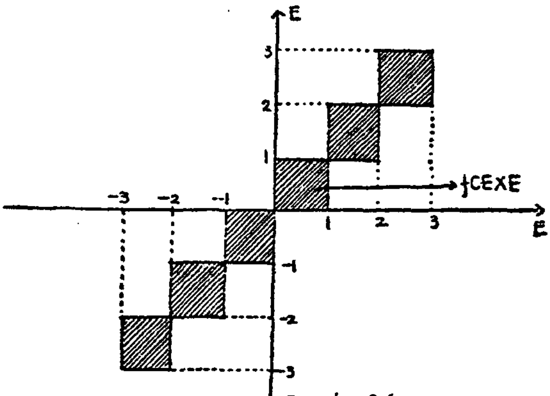
$A \times (B \cap C) = \{(x, y) \mid x \in A,\ y \in B \cap C\} = \{(x, y) \mid x \in A,\ y \in B,\ y \in C\}$
$= \{(x, y) \mid (x, y) \in A \times B,\ (x, y) \in A \times C\} = (A \times B) \cap (A \times C)$

3, 5. example Let R be the relation $<$ from$A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$,
that is, $(a, b) \in R$ if and only if $a < b$,
then $R = \{(a, b) \mid (a, b) \in R \subset A \times B,\ a < b\} = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$



Dom R=A, Ran R=B− {1} = {3, 5}

3, 6. example Let f be the relation from the set of real number E to E, defined by if both $x\in(n, n+1)$ and $y\in(n,n+1)$ for some integer n where $(n, n+1) = \{a \mid n\leqq a\leqq n+1\}$, then relation f consists of the shaded squares below.



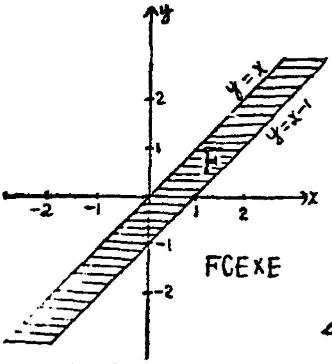
Graph of f

3, 7. example Let F be the relation in E×E by $(x, y)\in$F if and only if $0\leqq x-y\leqq 1$, then

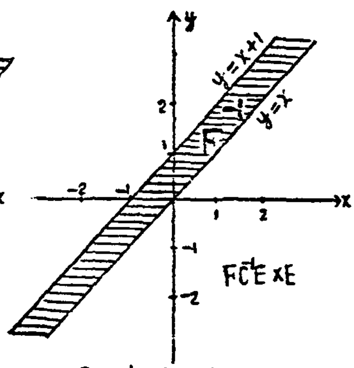
$F=\{(x, y) \mid x, y\in E, y\leqq x,\ y\geqq x-1\}$

$F^{-1}=\{(y, x) \mid x, y\in E,\ y\leqq x,\ y\geqq x-1\} = \{(x, y) \mid x, y\in E,\ y\geqq x,\ y\leqq x+1\}$

Futhermore $\{(x, y) \mid y=x+1\}$ is a relation.



Graph of F



Graph of F^{-1}

3, 8. (Def) A relation **R** in A (that is, R⊂A×A) is called an **equivalence relation** if:

(1) relexive, For each $a\in$A, $(a, a)\in$R

(2) symmetric, $(a, b)\in R\Rightarrow(b, a)\in$R

(3) transitive, $(a,b)\in R$, $(b,c)\in R \Rightarrow (a,c)\in R$

Here, if $(a,b)\in R$, we say that a and b are equivalent.

Let R be an equivalence relation in A. For each $a\in A$, the subset $R_a=\{b\mid(a,b)\in R\}\subset A$ is called the **R-equivalence class** of a.

The main theorem on equivalence relations is a consequence of following Lemma.

3, 9. Lemma Let R be an equivalence relation in A. then:

(1) $\cup\{R_a\mid a\in A\}=A$ (2) If $(a,b)\in R$, then $R_a=R_b$

(3) If $(a,b)\overline{\in} R$, then $R_a\cap R_b=\phi$

Proof (1) For each $a\in A$, because $(a,a)\in A$, we have $a\in R_a$, and therefore $A\subset\cup\{R_a\mid a\in A\}$. It follows $A=\cup\{R_a\mid a\in A\}$

(2) If $(a,b)\in R$ (that is $(b,a)\in R$)

$x\in R_a \Rightarrow \left.\begin{matrix}(x,a)\in R\\(a,b)\in R\end{matrix}\right\}\Rightarrow(x,b)\in R\Rightarrow x\in R_b$, this showing $R_a\subset R_b$.

$x\in R_b \Rightarrow \left.\begin{matrix}(x,b)\in R\\(b,a)\in R\end{matrix}\right\}\Rightarrow(x,a)\in R$ $x\in R_a$, this proving that $R_b\subset R_a$, Consequently $R_c=R_b$.

(3) Assume $R_a\cup R_b\neq\phi$, then there is k such that $k\in R_a\cup R_b$, we have $(k,a)\in R$ dan $(k,b)\in R$, so that symmetry and transitivity of R give $(a,b)\in R$, showing that proof of (3) is complete.

3, 10. (Def) If $\{A_\alpha\mid\alpha\in\mathfrak{A}\}$ is a covering of X, that is, $\cup\{A_\alpha\mid\alpha\in\mathfrak{A}\}=X$ and if $A_\alpha\subset A_\beta=\phi$ whenever $\alpha\neq\beta$, then the family $\{A_\alpha\mid\alpha\in\mathfrak{A}\}$ is called a **Partition** of X.

3, 11. Lemma If $\{A_\alpha\mid\alpha\in\mathfrak{A}\}$ is a partition of X, then there existan equivalence relation defined by $x\in A_\alpha$ and $y\in A_\alpha$ for some $\alpha\in\mathfrak{A}$ if and only if $(x,y)\in R$.

Proof (1) For each $x\in X$, because $\{A_\alpha\mid\alpha\in A\}$ is a partition of X, there is a $\alpha\in\mathfrak{A}$ such that $x\in A_\alpha$. Thus $(x,x)\in R$, this means that R is reflexive.

(2) $(x,y)\in R\Longleftrightarrow x\in A_\alpha$ for some $\alpha\in A\Longleftrightarrow(y,x)\in R$, that is, R is symmetric.

(3) $(x,x)\in R$, $(y,z)\in R\Longleftrightarrow(x\in A_\alpha,\ y\in A_\alpha)$ and $(y\in A_\beta,\ z\in A_\beta)$, Since family $\{A_\alpha\}$ is a partition, $\alpha=\beta$.

$\Longleftrightarrow x,y,z\in A_\alpha\Rightarrow(x,z)\in R$: R is transitive.

Lemma 3, 9∼11. give the following theorem

3, 12. theorem R is an equivalence relation in A if and only if there is a partition $\{A_\alpha\mid\alpha\in\mathfrak{A}\}$. Futhermore the set A_α is precisely the R-equivalence classes $\{R_a\mid(a,b)\in R,\ a\in A\}$

Each element of an R-equivalense class R_a is called a representative of R_a. With each equivalence classes we constructed a new set according to the following.

3, 13. (Def) Let A have an equivalence relation R. The set whose elements are the R-equivalence classes is called the **quotient set** of A by R and is written **A/R.**

3, 14. example Let R be the relation in I, the set of integers, defined by $(x,y)\in R\Longleftrightarrow x\equiv y \pmod 3$ which reads "x is congruent to y modulo 3" and which means "$x-y$ is divisible by 3". Show that R is an equivalence relation and quotient set.

Proof (1) $x-x=0=3\cdot 0$, $0\in I$, Hence $x\equiv x \pmod 3)\Longleftrightarrow(x,x)\in R$. (reflexive)

(2) $(x,y)\in R\Longleftrightarrow x\equiv y\pmod 3)\Rightarrow x-y=3k$, $k\in I\Rightarrow y-x=3(-k)$, $-k\in I\Rightarrow y\equiv x\pmod 3)$

$\Longleftrightarrow (y, x) \in R$ (Symmetric)

(3) $(x, y) \in R,\ (y, z) \in R \Rightarrow x \equiv y \pmod 3,\ y \equiv z \pmod 3 \Rightarrow x-y=3m,\ y-z=3n,\ m, n \in I$

$\Rightarrow x-z=(x-y)+(y-z)=3m+3n=3(m+n),\ m+n \in I$

$\Rightarrow x \equiv z \pmod 3 \Longleftrightarrow (x, z) \in R$ (transitive)

Therefore, by 3.12 theorem, there is exactly the quotient set $I/R=\{R_0, R_1, R_2\}$ such that

$R_0=\{\cdots, -6, -3, 0, 3, 6, \cdots\}=\cdots=R_{-6}=R_{-3}=R_3=R_6=\cdots$

$R_1=\{\cdots, -5, -2, 1, 4, 7, \cdots\}=\cdots=R_{-5}=R_{-2}=R_4=R_7=\cdots$

$R_2=\{\cdots, -4, -1, 2, 5, 8, \cdots\}=\cdots=R_{-4}=R_{-1}=R_5=R_8=\cdots$

3, 15. example: Let X be the set of all students of our school.

If we define a relation R in X such that

$(x, y) \in R \Longleftrightarrow$ First name of x and y are same or $x=y$.

then R is an equivalence relation and hence

$X/R=\{R_{kim},\ R_{lee},\ R_{park},\ R_{jung}\}$.

But "x is a friend of y" is not transive.

3, 16. example

Let $A=\{1, 2, 3, 4\}$ and $R=\{(1, 1)(1, 2)(2, 1)(2, 2)(3, 3)(4, 4)\}$

then R is an equivalence relation and the quotient set A/R is the set $\{\{1, 2\}, 3, 4\}$.

3, 17. example

Let **X** be a set and **I** is the unit interval $[0, 1]$, then the set $[\mathbf{X} \times \{1\}] \times [X \times \{-1\}]$ is an equivalence relation in $\mathbf{X} \times \mathbf{I}$ and the quotient set $\mathbf{X} \times I/X \times \{1\}$ is obtained by $\mathbf{X} \times \mathbf{I}$ by pinching $\mathbf{X} \times \{1\}$ to a single point.

4. Functions and Continuous functions.

4, 1. (Def) Let f be a relation and let A be a set. We define the **image of A under f** to be the set

$f(A)=\{y \mid (x, y) \in f$ for each $x \in A\}$.

The inverse image of A under f is the set $f^{-1}(A)$. And we define f is a function if and only if $(x, y) \in f$ and $(x, z) \in f$ imply $y=z$.

4, 2. (Def) Let **X** be a set. A topology in **X** is a family τ of subsets of **X** that satisfies:

(1) Each union of members of τ is also a member of τ

(2) Each finite intersection of members of τ is also a member of τ

(3) ϕ and X are members of τ

A couple (X, τ) consisting of a set X and a topology τ in X is called a **topological space (or space)**.

The members of τ are called the **"open sets"** of the topological space (X, τ), the complement of the open set is called **"closed set"**. For some $x \Rightarrow X$, any open set containing x is called a neighborhood (written nbd or U_x) of x.

4, 3. example Let X be a set, $\tau=\{\phi, X\}$. This topology, in which no set other than ϕ and X is open, is called the indiscrete topology. $\tau=\mathfrak{P}(X)$ is called discrete topology. In the sense different topologies in a set X give different organization of the points (elements) of X.

4, 4. theorem A set is open if and only if it contains a neighborhood of each of its points.

Proof U= {B|B is open subset of A} is clealy open and is a subset of A(U⊂A). If A contains a neighborhood of each of its points, then each point x of A belongs to some open subset of A and hence $x \in U$. In this case A=U and therefore A is open. On the other hand, if A is open it contains a neighborhood (i,e,A) of each of its points.

4, 5. theorem: Euclidian topology on real line E

Let E be the set of real numbers, a set G⊂E is "open" if for each $x \in G$ there is an $\gamma > 0$ such that the symmetric open interval $B(x, \gamma) = (x-\gamma,\ x+\gamma) = \{y \mid |y-x| < \gamma\} \subset G$. Then the family τ of sets declared "open" is a topology in the set E.

Proof (1) If each member of $\{G_\alpha \mid \alpha \in A\}$ is "open", since

$x \in \bigcup_\alpha G_\alpha \Rightarrow$ there is an α such that $x \in G_\alpha (\exists \alpha : x \in G_\alpha)$

$\Rightarrow \exists \gamma > 0 : B(x : \gamma)(G_\alpha \subset \bigcup_\alpha G_\alpha)$, each union of members of τ is also a member of τ.

(2) If $G_1, G_2, \cdots\cdots, G_n$ are "open", because $x \in \bigcap_{i=1}^n G_i \Rightarrow$ (for all $i : x \in G_i$)

$\Rightarrow$ (for all i $\exists \gamma_i > 0 : B(x : \gamma_i) \subset G_i) \Rightarrow (B(x : \min(\gamma_1, \gamma_2, \cdots\cdots, \gamma_n)) \subset \bigcap_{i=1}^n G_i)$,

each finite intersection of members of τ is a also a member of τ.

(3) is trivial. which completes the proof.

This topology τ is called the Euclidian (usual) topology, the topological space (E, τ) is called Euclidian space, Each open interval $U = \{x \mid a < x < b\} = (a, b)$ is a open set in Euclidian space (E, τ). Because for each $x \in U \Rightarrow (E\tau : B(x : \mathrm{Min}(x-a, b-x)) \subset U) \Rightarrow U \in \tau$.

It is simple to see that τ can also be described more directly as the family of all unions of open intervals.

4, 6. [Def] Let X and Y be two topological spaces, and let f be a function on X into Y, We define f is **continuous** if and only if the inverse image of each open set in Y under f is open in X.

4, 7. theorem Function f is continuous on X to Y if and only if for each $x \in X$ and each nbd $W_{f(x)}$ in Y, there exist a nbd V_x in X such that $f(V_x) \subset W_{f(x)}$.

Proof (only if) For each nbd $W_{f(x)}$ in Y there exist a nbd $V_{f(x)} \subset W_{f(x)}$ because $W_{f(x)}$ is open. Thus, since $f^{-1}(V_{f(x)}) = U_x$ is open in X, there is a nbd U_x such that $f(U_x) = V_{f(x)} \subset W_{f(x)}$.
(if) Let U is open in Y, then for each $x \in f^{-1}(U)$, there exist a nbd V_x such that $f(V_x) \subset U$, therefore $x \in V_x \subset f^{-1}(U)$, it showing that $f^{-1}(U)$ is open in X. Hence f is continuous.

4, 8. [Def] An $f : X \to Y$ is **continuous at $x_0 \in X$** if 4, 7 theorem is satisfied at x_0, that is, for each nbd $W_{f(x_0)}$, there is a nbd V_x in X such that $f(V_x) \subset W_{f(x)}$.

4, 9. Remark Let $f : E \to E$, that is, a real valued function on the set of real number. The continuity at $x_0 \in E$ is simply to usual notion encountered in real analysis. For: By 4, 5 and 4, 8, f is continuous at x_0 if for each open interval (that is open set) $W_{f(x_0)} = (f(x_0) - \epsilon,\ f(x_0) + \epsilon)$, there is symmetric open interval $V_{x_0} = B(x_0 : \delta)$ mapped into it by f, that is, for each $\epsilon > 0$ there exist $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$, written $\forall \epsilon > 0,\ \exists \delta > 0 : |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$, because $f(V_{x_0}) \subset W_{f(x_0)}$.

4.10. theorem Let f and g are continuous at x_0 on E into E, that is, $\forall\epsilon>0$,

$\exists\delta>0 : |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\epsilon$

$\exists\delta>0 : |x-x_0|<\delta \Rightarrow |g(x)-g(x_0)|<\epsilon$, respectively,

then ① $f(x)+g(x)$ ② $f(x)\cdot g(x)$

③ $c\cdot f(x)$, for any real number c ④ $\frac{1}{f(x)}$ provided $f(x_0)\neq 0$

are continiuous at x_0.

Proof (1) $\forall\epsilon>0$, there exist δ_1, δ_2 such that

$|x-x_0|<\delta_1$ imples $|f(x)-f(x_0)|<\frac{\epsilon}{2}$

$|x-x_0|<\delta_2$ implies $|g(x)-g(x_0)|<\frac{\epsilon}{2}$.

Therefore, if we take $\delta=\mathrm{Min}(\delta_1, \delta_2)$ implies

$$|f(x)+g(x)-(f(x_0)+g(x_0))|\leqq|f(x)-f(x_0)|+|g(x)-g(x^0)|<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon$$

This proves (1)

(2) we use the identity

$$f(x)g(x)-f(x_0)g(x_0)=(f(x)-f(x_0))(g(x)-g(x_0))$$
$$+f(x_0)(g(x)-g(x_0))+g(x_0)(f(x)-f(x_0)).$$

Given $\epsilon>0$,

$\exists\delta_1 : |x-x_0|<\delta_1 \Rightarrow |f(x)-f(x_0)|<\sqrt{\frac{\epsilon}{3}}$

$\exists\delta_2 : |x-x_0|<\delta_2 \Rightarrow |g(x)-g(x_0)|<\sqrt{\frac{\epsilon}{3}}$

$\exists\delta_3 : |x-x_0|<\delta_3 \Rightarrow |g(x)-g(x_0)|<\frac{\epsilon}{3|f(x_0)|}$

$\exists\delta_4 : |x-x_0|<\delta_4 \Rightarrow |f(x)-f(x_0)|<\frac{\epsilon}{3|g(x_0)|}$

therefore, for each $x\in B(x_0;\delta)$ such that $\delta=\mathrm{Min}(\delta_1, \delta_2, \delta_3, \delta_4)$

$$|f(x)g(x)-f(x_0)g(x_0)|\leqq|f(x)-f(x_0)|\cdot|g(x)-g(x_0)|$$
$$+|f(x_0)|\cdot|g(x)-g(x_0)|+|g(x_0)|\cdot|f(x)-f(x_0)|<\sqrt{\frac{\epsilon}{3}}\cdot\sqrt{\frac{\epsilon}{3}}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon$$

If $f(x_0)\cdot g(x_0)=0$, the proof is more simple.

(3) Given $\epsilon>0$, $\exists\delta>0 : |x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\frac{\epsilon}{|C|}$.

Thus, $\forall x\in B(x_0 : \delta)$

$$|cf(x)-cf(x_0)|\leqq|c|\,|f(x)-f(x_0)|-f(x_0)|<|c|\cdot\frac{\epsilon}{|c|}=\epsilon$$

If $c=0$, the function cf is constant.

(4) Given $\epsilon>0$, if we take k such that $|f(x_0)|>k>0$,

then, $\exists\delta_1 : |x-x_0|<\delta_1 \Rightarrow |f(x)|>k>0$

and $\exists\delta_2 : |x-x_0|<\delta_2 \Rightarrow |f(x)-f(x_0)|<k|f(x_0)|\epsilon$.

Hence, let $\delta=\mathrm{Min}(\delta_1, \delta_2)$, then $|x-x_0|<\delta \Rightarrow$

$$\left|\frac{1}{f(x)}-\frac{1}{f(x_0)}\right|=\left|\frac{f(x)-f(x_0)}{f(x)f(x_0)}\right|\leqq\frac{1}{|f(x)|\,|f(x_0)|}|f(x)-f(x_0)|<\frac{1}{k|f(x_0)|}\cdot k|f(x_0)|\epsilon=\epsilon.$$

this implies $\frac{1}{f(x)}$ is continuous at x_0.

Since, by 4.10 theorem (3), (4), $\frac{g(x)}{f(x)}=g(x)\cdot\frac{1}{f(x)}$ is continuous at x_0

4, 11. example

$f=\left\{(x,y)\,|\,y=\sin\frac{1}{x}\right\}$ is not continuous at $x=0$. For: for each $\alpha>0$, there exist $x=\frac{1}{n\pi}<\delta$ such that $\sin\frac{1}{x}=1$. therefore, there is not open set $(-\alpha,\alpha)$ such that

$f[(-\alpha,\alpha)]\subset\left(-\frac{1}{2},\ \frac{1}{2}\right)$ in range space.

4, 12. theorem Let$(X, \mathfrak{T})$ is a topological space and let A is a subset of X, then $\mathfrak{T}=\{B\cup A\,|\,B\in\tau\}$ is a topology for A and this called relative topology on X.

Proof (1) Let $B_\alpha\in\tau$, $B_\alpha\cap A\in\mathfrak{T}$ for each α, then, by equality $\bigcup_\alpha(B_\alpha\cap A)=(\bigcup_\alpha B_\alpha)\cap A\Rightarrow\bigcup_\alpha B_\alpha\in\mathfrak{T}$.

(2) $\bigcap_{\alpha=1}^{n}(B_\alpha\cap A)=(\bigcap_{\alpha=1}^{n}B_\alpha)\cap A\Rightarrow\bigcap_{\alpha=1}^{n}B_\alpha\in\mathfrak{T}$

(3) $\phi\cap A=\varphi$, $X\cap A=A$ imply ϕ, $X\in\mathfrak{T}$

4, 13. [Def] We deffne (A, F) is a **subspace** of $(X\tau)$ in 4.12 theorem

4, 14. [Def] A space **X** is **Connected** if it is not the union of two nonempty disjoint open set. A subset $B\subset$**X** is connected as a subspace.

For example, Let $\mathbf{X}=\{0,1\}$ $\tau=\{\{0\},\phi,x\}$ $\mathfrak{P}(x)$,

then topological space (x,τ) is conected but $(X,\ \mathfrak{P}(x))$ is not connected.

4, 15. theorem; the only connected subset of Euclidian space E having more than one point are E and intervals (open, closed, or the of half open)

Proof It is uffice to show that, if $x\in A$, $y\in A$ and $x<z<y$ then $z\in A$ if and only if A is connected.

(If) If $x\in A$, $y\in A$ and $x<z<y$ then $z\overline{\in}A$ then

$A=(\{\alpha|\alpha<z\}\cap A)\cup(\{\alpha|\alpha>z\}\cap A)$. It follows A is not connected by 4.12, 13, 14.

(only if) Suppose A is not connected, then there exist elements $x\in A$, $y\in A$ with $x<y$, and there exist disjoint ope set B,C, in A such that $x\in B$, $y\in C$, and $A\subset B\cup C$

Let $S=B\cap[x,y]$ and let z be the least upper bound of S. Since $x\in B$, B is open, we have $x<z$, thus if we had $z\in C$, the fact that C is open would show that z is not least upper bound of S. Hence $z\overline{\in}C$. Since $A\subset B\cup C$ it follows that $z\overline{\in}A$, which completes the proof.

4, 16. Theorem Let f is continuous function on X into Y and let X is connected then $f(X)\subset Y$ is also connected.

Proof If $f(X)$ is not connected, there are disjoint open sets U and V in Y, both of which intersect $f(X)$, such that $f(X)\subset U\cap V$. Since f is continuous, the set $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. they are clearly disjoint and nonempty, and their union is X. this means that X is not connected in contradiction to the hypothesis.

4, 17. Theorem Let f be a continuous real valued function on a connected space X. If $f(x)=a$, $f(x')=b$ then $[a,b]\subset f(X)$. That is, for each c such that $a\leqq c\leqq b$, there is an z with $f(z)=c$.

Proof Since the image of a connected set under continuous function is also connected by 4.16, $f(X)\subset E$ is connected. Therefore $f(X)$ is an interval according to 4.15. Thus, if $f(X)=a$, $f(X')=b$, we have $[a,b]\subset f(x)$.

This theorem is called the generalization of intermediate values theorem. (P.11로 계속)